Goals: 1. Learn basic mathematical techniques for solving EM problems.

2. Study some representative problems in depth, and methods for solving a variety of problems.

3. Develop intuition & ability to visualize field solutions.

⇒ * Develop skill of mathematical analysis of physical problems ... *

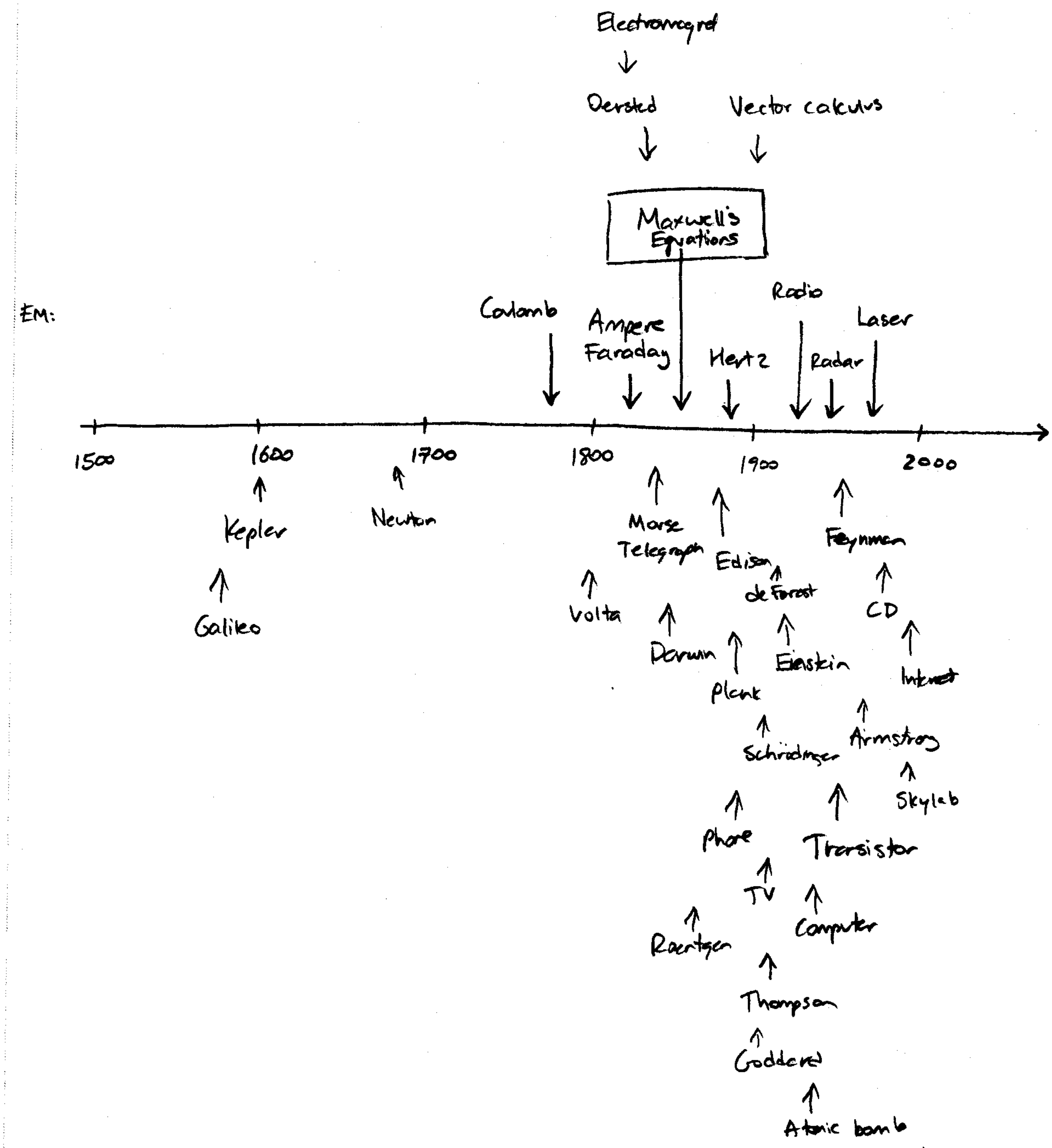
Topics:

Media
Green's functions
  Radiation, antennas
Theorems & principles
Scattering
High frequency asymptotics

Tips:

- This is a grad class - enjoy it!
- Grad courses should tie things together into a "bigger picture"
- Don't worry quite as much about grades...
- focus on research

TIMELINE:


Electromagnet
Oersted
Vector calculus
Maxwell's Equations
EM:
Coulomb
Ampere
Faraday
Hertz
Radio
Radar
Laser
1500
1600
1700
1800
1900
2000
Kepler
Galileo
Newton
Volta
Morse Telegraph
Darwin
Edison
de Forest
Plank
Einstein
Schrödinger
Phone
TV
Roentgen
Thompson
Goddard
Atomic bomb
Transistor
Computer
Feynman
CD
Internet
Armstrong
Skylab

Today, a vast array of technologies have come from the field of electromagnetics. Here are a few:

Microwave systems
- Waveguides, microstrip
- Communication links
- Radar
- GPS, sattelite repeaters
- Wireless networks

Antennas
- Arrays
- Planar antennas

Optics
- fibers
- lithography

Remote sensing
- Sattelite imaging
- Deep space probes
- Radio astronomy

Biomedical
- Microwave imaging
- MRI
- Infrared, x-ray sensing

High speed digital systems

Reading: 1.1

Topics: Vector calculus
Maxwell's Equations

Maxwell's Equations:

$\vec{E}$: V/m
$\vec{H}$: A/m
} Field intensity (1-forms)

$\vec{D}$: C/m^2
$\vec{B}$: Wb/m^2
} Flux density (2-forms)

Sources: $\vec{J}$: A/m^2 Current density (2-form)
ρ : C/m^3 Charge density (3-form)

$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (Faraday's law)

$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$ (Ampere's law)

$\nabla \cdot \vec{D} = \rho$ (Gauss's law for electric flux)

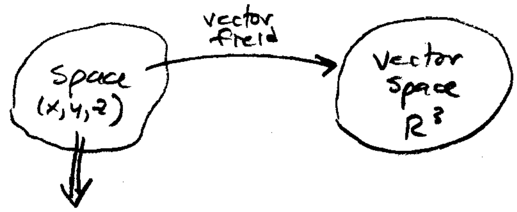
$\nabla \cdot \vec{B} = 0$ (" " magnetic flux)

$\vec{D} = \epsilon_0 \vec{E}$
$\vec{B} = \mu_0 \vec{H}$
} Constitutive relations (→ material models)
ϵ_0 : F/m, μ_0 : H/m

$\vec{J} = \sigma \vec{E}$ } Ohm's law (point form) - induced current
σ : S/m

Vector: $A_1\hat{x} + A_2\hat{y} + A_3\hat{z} = \vec{A} \in$ Vector space (in this case, R^3)

Vector field: map from a space to a vector space

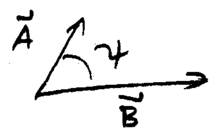


$$\vec{A} = A_1(x,y,z)\,\hat{x} + A_2(x,y,z)\,\hat{y} + A_3(x,y,z)\,\hat{z}$$

Note: The space (x,y,z) is also a vector space – R^3 !?!
$\Rightarrow \vec{r} = x\hat{x} + y\hat{y} + z\hat{z} \leftrightarrow (x,y,z)$ ...

Dot product or inner product: map from $R^3 \times R^3 \rightarrow R$

$$\vec{A}\cdot\vec{B} = A_1B_1 + A_2B_2 + A_3B_3 = |\vec{A}||\vec{B}|\cos\psi$$
$$= AB\cos\psi \qquad (A \equiv |\vec{A}|)$$



Cross product: map from $R^3 \times R^3 \rightarrow R^3$

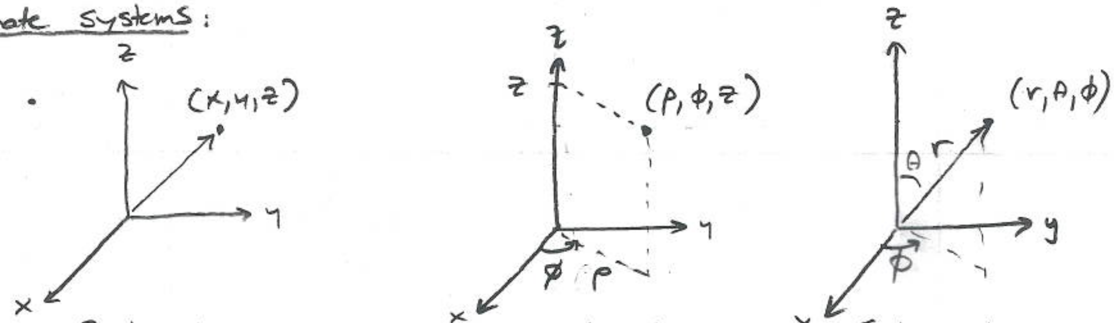
$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}, \quad \hat{x}\times\hat{y} = -\hat{y}\times\hat{x} = \hat{z}$$
etc...

Gradient: map, $R \rightarrow R^3$, $\nabla f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$

Curl: map, $R^3 \rightarrow R^3$, $\nabla\times\vec{A} = (\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z})\times\vec{A}$ ...

Divergence: map, $R^3 \rightarrow R$, $\nabla\cdot\vec{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$

Coordinate systems:



Rectangular | Cylindrical | Spherical

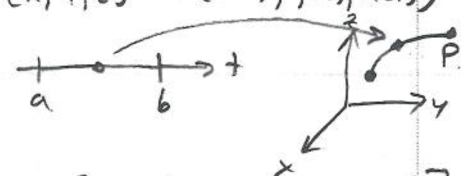
$dV = dx\,dy\,dz \equiv d\vec{r}$ | $dV = \rho\, d\rho\, d\phi\, dz$ | $dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$

$\hat{x}, \hat{y}, \hat{z}$ | $\hat{\rho}, \hat{\phi}, \hat{z}$ | $\hat{r}, \hat{\theta}, \hat{\phi}$

Line integral:

Path: map from $\mathbb{R}$ to $\mathbb{R}^3$

$(x,y,z) = (f(t), g(t), h(t))$



$$\int_P \vec{A}\cdot d\vec{l} = \int_a^b \Big[A_1(f(t),g(t),h(t))\,\hat{x} + \cdots\Big]\cdot d\Big[f(t)\,\hat{x} + g(t)\,\hat{y} + h(t)\,\hat{z}\Big]$$

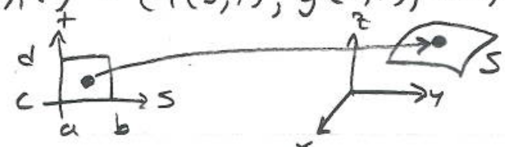
$$= \int_a^b \Big[\quad \text{"} \quad\Big]\cdot\Big[\frac{\partial f}{\partial t}dt\,\hat{x} + \frac{\partial g}{\partial t}dt\,\hat{y} + \frac{\partial h}{\partial t}dt\,\hat{z}\Big]$$

$$= \int_a^b \Big[\text{function of } t\Big]\,dt \Rightarrow \text{calculus} \ldots$$

Surface integral:

Surface: map from $\mathbb{R}^2$ to $\mathbb{R}^3$

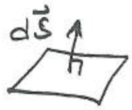
$(x,y,z) = (f(s,t), g(s,t), h(s,t))$



$$\int_S \vec{A}\cdot d\vec{s} = \int_c^d\int_a^b \vec{A}(f(s,t), g(s,t), h(s,t))\cdot\Big[\frac{\partial\vec{r}}{\partial s}\times\frac{\partial\vec{r}}{\partial t}\Big]ds\,dt$$

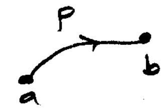
$$= \int_c^d\int_a^b \vec{A}\cdot\Big[\frac{\partial f}{\partial s}\hat{x} + \frac{\partial g}{\partial s}\hat{y} + \frac{\partial h}{\partial s}\hat{z}\Big]\times\Big[\frac{\partial f}{\partial t}\hat{x} + \frac{\partial g}{\partial t}\hat{y} + \frac{\partial h}{\partial t}\hat{z}\Big]ds\,dt$$

$$= \int_c^d\int_a^b \Big[\text{function of } s,t\Big]\,ds\,dt \Rightarrow \text{calculus} \ldots$$



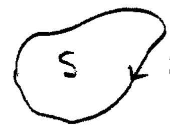
<u>Fund. thm. of calulus</u>:

$$\int_P \nabla f \cdot d\vec{l} = f(b) - f(a)$$



<u>Stokes theorem</u>:

$$\iint_S (\nabla \times \vec{A}) \cdot d\vec{s} = \oint_{\partial S} \vec{A} \cdot d\vec{l}$$



<u>Divergence theorem</u>:

$$\iiint_V \nabla \cdot \vec{A} \, dV = \oiint_{\partial V} \vec{A} \cdot d\vec{s}$$

<u>Integral forms</u>:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\int_S d\vec{s} \cdot (\nabla \times \vec{E}) = -\int_S d\vec{s} \cdot \frac{\partial \vec{B}}{\partial t} \qquad \text{(integrate both sides)}$$

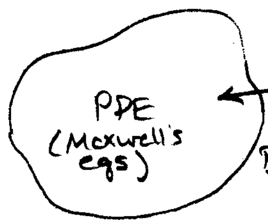
$$\underbrace{\oint_{\partial S} \vec{E} \cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}}_{\text{Integral form of Faraday's law ...}} \qquad \text{(Stokes theorem)}$$

... similarly for other 3 laws ...

There are many different types of EM problems ... this can be confusing. A way to bring them together is to view them as

Boundary value problems (BVP)

$$\text{BVP} = \begin{cases} ① \text{ Partial differential equation (PDE)} \\ \qquad + \\ ② \text{ Boundary condition (BC)} \end{cases}$$



unique solution

BC = boundary values of $\vec{E}, \vec{H}$

Modes:

One complication is that we often solve EM problems without taking into account specific values of the sources $\vec{J}$ and ρ, which in the mathematical sense are forcing functions. In this case, instead of finding a unique solution, we obtain a family of possible solutions (modes) which can be driven by a given source ...

Impressed vs. Induced Current

$\vec{J}_{induced}$ = current driven by $\vec{E}, \vec{H}$ by Lorentz force law ... bulk model: $\vec{J}_{induced} = \sigma \vec{E}$
↳ (part of PDE...)

$\vec{J}_{impressed}$ = current imposed by an external agent... unaffected by its own $\vec{E}, \vec{H}$ fields } engineering approximation
↳ (forcing function)

Lorentz Force Law

Electromagnetic fields exert a force on charged particles:

$$\vec{F} = q\vec{E} + q\vec{v}\times\vec{B}$$

If the charges are restrained by other mechanical forces, this leads to dissipation of energy.

Maxwell's equations + Lorentz force law + Newtons law ($\vec{F} = m\vec{a}$)

= "Electro<u>dynamics</u>"

Poynting's theorem

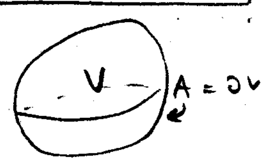
$$\nabla\cdot(\vec{E}\times\vec{H}) = \vec{H}\cdot(\nabla\times\vec{E}) - \vec{E}\cdot(\nabla\times\vec{H})$$
$$= \vec{H}\cdot\left(-\frac{\partial\vec{B}}{\partial t}\right) - \vec{E}\cdot\left(\frac{\partial\vec{D}}{\partial t} + \vec{J}\right)$$
$$= -\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} - \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} - \vec{E}\cdot\vec{J}$$

Poynting vector: $\vec{S} = \vec{E}\times\vec{H}$ (W/m²)

Theorem: $$\nabla\cdot\vec{S} = -\left[\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\right] - \vec{E}\cdot\vec{J}$$

Integral form: $$\underbrace{\oint_A \vec{S}\cdot d\vec{A}}_{\text{Energy flowing accross boundary of } V} = \underbrace{-\frac{1}{2}\iiint_V dv\,\frac{\partial w}{\partial t}}_{\text{Rate of change of energy stored in } V} - \underbrace{\iiint_V dv\,\vec{E}\cdot\vec{J}}_{\text{Energy suppled/dissipated by currents}}$$

$\boxed{\text{Conservation of Energy}}$



$$\frac{\partial w}{\partial t} = 2\left[\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\right]$$ = time rate of change of energy stored by E, H at a point

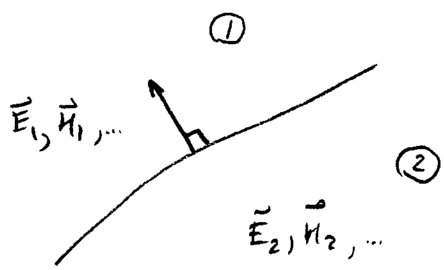
Boundary conditions are very important in EM theory. The most common set of B.C.'s are the interface conditions

$$\hat{n} \times (\vec{E}_1 - \vec{E}_2) = 0$$

$$\hat{n} \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_s \quad (A/m) \quad \text{- surface current density}$$

$$\hat{n} \cdot (\vec{D}_1 - \vec{D}_2) = \rho_s \quad (C/m^2) \quad \text{- surface charge density}$$

$$\hat{n} \cdot (\vec{B}_1 - \vec{B}_2) = 0$$

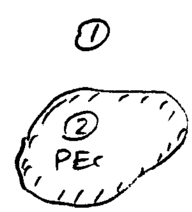


The boundary between regions ① and ② will often coincide with a physical change in materials, such as a dielectric medium or conductor.

PEC

The "perfect electric conductor" is an idealization but also very useful. Since $\sigma \to \infty$, $\vec{J} = \sigma \vec{E}$ tells us that $\vec{E} = 0$ in a PEC object. Similarly, $\vec{H} = \vec{D} = \vec{B} = 0$.

Thus, at the surface of a PEC object,



$$\hat{n} \times \vec{E} = 0$$

$$\hat{n} \times \vec{H} = \vec{J}_s$$

$$\hat{n} \cdot \vec{D} = \rho_s$$

$$\hat{n} \cdot \vec{B} = 0$$

Time Harmonic fields

EM sources are often narrowband, so we use a single frequency analysis. In such cases, phasor fields are convenient:

$$\vec{E}(\vec{r},t) = \mathrm{Re}\{\underbrace{\vec{E}(\vec{r})}_{\text{phasor}}e^{-i\omega t}\} \qquad (1)$$

Note: most EM texts use $e^{j\omega t}$ instead of $e^{-i\omega t}$. Kong uses a different convention that is common in physics.

Also, since the same symbol is used for the time-varying field and its phasor representation, context must be used to distinguish them if the arguments are suppressed.

With phasors, $\frac{\partial}{\partial t} \rightarrow -i\omega$, so Maxwell's equations become

$$\left.\begin{aligned} \nabla\times\vec{E} &= i\omega\vec{B} \\ \nabla\times\vec{H} &= -i\omega\vec{D} + J \end{aligned}\right\} \text{ Phasor form}$$

and so forth.

The complex Poynting vector is

$$\vec{S}(\vec{r}) = \vec{E}(\vec{r}) \times \vec{H}(\vec{r})^*$$

By using the definition (1) and integrating over one period, we find that the time-average power flux is

$$\vec{S}_{av}(\vec{r}) \equiv \frac{1}{T}\int_0^T \vec{S}(\vec{r},t)\,dt$$

$$= \frac{1}{2}\mathrm{Re}\{\vec{E}(\vec{r}) \times \vec{H}(\vec{r})^*\}$$

<u>Wave Equation</u>

By taking the curl of Faraday's law, substituting Ampere's law, using the identity $-\nabla\times\nabla\times\vec{A} + \nabla\nabla\cdot\vec{A} = \nabla^2\vec{A}$, and Gauss's law, we obtain

$$(\nabla^2 + k^2)\vec{E} = 0 \qquad (1)$$

when $k = \omega\sqrt{\mu\epsilon}$. The general solution to this PDE (Helmholtz Equation) is

$$\vec{E}(\vec{r}) = \underset{\substack{\uparrow \\ \text{constant} \\ \text{vector}}}{\vec{E}_0}\, e^{i\vec{k}\cdot\vec{r}}$$

where $\vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z}$. A relationship between the constants k_x, k_y, and k_z can be found by substituting into the Helmholtz equation. This is the dispersion relation

$$k_x^2 + k_y^2 + k_z^2 = \omega^2\mu\epsilon = k^2$$

Physically, the wave vector $\vec{k}$ gives the direction of propagation of the wave solution.

One final constraint can be found by substituting $\vec{E}(\vec{r})$ into Gauss's law:

$$\nabla\cdot(\epsilon_0\vec{E}) = 0$$

$$\nabla\cdot\vec{E} = 0$$

$$i\vec{k}\cdot\vec{E} = 0$$

so that $\vec{k} \perp E_0$. Note that (1) has solutions which do not satisfy this constraint.