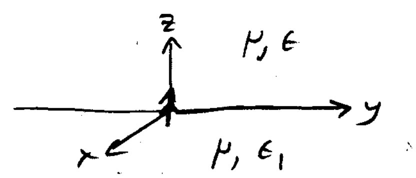
We consider a vertical magnetic dipole (VMD) on a half-space medium:



From the layered media treatment, the z component of the magnetic field is of the form

$$H_z = \sum_m \int dk_\rho \left[C_m e^{ik_z z} + D_m e^{-ik_z z} \right] H_m^{(1)}(k_\rho \rho)\, C_m(\phi)$$

Because the dipole is vertical, the fields are ϕ-independent, so that $C_m = 0$, $m \neq 0$, and $C_0 =$ constant. Since the fields for $z > 0$ are traveling away from the dipole by causality, $D_m = 0$. Thus, we have

$$H_z = \int dk_\rho\, C_0(k_\rho)\, e^{ik_z z} H_0^{(1)}(k_\rho \rho)$$

~~The constant~~ C_0 can be found from the field due to the VMD in single homogeneous region:

$$H_z = \hat{z} \cdot \left\{ i\omega\epsilon \left[\bar{\bar{I}} + \frac{\nabla\nabla}{k^2} \right] \int d\bar{r}' \frac{e^{ik|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|} \bar{M}(\bar{r}') \right\} \quad \text{(single region)}$$

where the magnetic current is $\bar{M}(\bar{r}) = \hat{z}\, m\ell\, \delta(\bar{r})$, which is a magnetic dipole of magnetic current m and length ℓ. (On p. 520, it is shown that this source is equivalent away from the source to a small current loop of area A and current I if $m\ell = -i\omega\mu I A$). The field becomes

$$H_z = -i\omega\epsilon \left[1 + \frac{1}{k^2}\frac{\partial^2}{\partial z^2} \right] \frac{e^{ikr}}{4\pi r}\, m\ell$$

$$= -i\omega\epsilon \left[1 + \frac{1}{k^2}\frac{\partial^2}{\partial z^2} \right] \frac{i}{8\pi} \int dk_\rho \frac{k_\rho}{k_z} H_0^{(1)}(k_\rho \rho)\, e^{ik_z|z|}\, m\ell$$

$$= +\frac{\omega\epsilon}{8\pi} m\ell \int dk_\rho \frac{k_\rho}{k_z} \left(1 - \frac{1}{k^2} k_z^2 \right) H_0^{(1)}(k_\rho \rho)\, e^{ik_z|z|}$$

$$= +\frac{\omega\epsilon}{8\pi} m\ell \int dk_\rho \frac{k_\rho^3}{k^2 k_z} H_0^{(1)}(k_\rho \rho)\, e^{ik_z|z|}$$

$$= + \frac{\omega\epsilon}{\omega^2\mu\epsilon}(-i\omega\mu IA)\int dk_\rho \, \frac{k_\rho^3}{k_z} H_0^{(1)}(k_\rho\rho)e^{ik_z|z|}$$

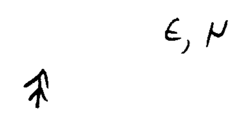
$$= \int dk_\rho \left[\frac{k_\rho^3(-iIA)}{k_z}\right] H_0^{(1)}(k_\rho\rho)e^{ik_z|z|}$$

$$= \int dk_\rho \left[-i\,\frac{IA\,k_\rho^3}{8\pi k_z}\right] H_0^{(1)}(k_\rho\rho)\, e^{ik_z|z|}$$

If we add a layer of permittivity ϵ_1 below the VMD, how does this change? We simply add the reflected wave due to the layer below the dipole:

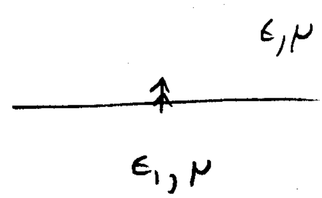
No half-space

$$H_z = \int dk_\rho \left[-i\,\frac{IA\,k_\rho^3}{8\pi k_z}\right] H_0^{(1)}(k_\rho\rho)e^{ik_z z}$$



With half-space

$$H_z = \int dk_\rho \left[-i\frac{IA\,k_\rho^3}{8\pi k_z}(1+R)\right] H_0^{(1)}(k_\rho\rho)e^{ik_z z}$$



where R is the reflection coefficient of the interface. The polarization is TE, since the electric field is horizontal, and the reflection coefficient becomes

$$R^{TE} = \frac{1-\frac{k_{1z}}{k_z}}{1+\frac{k_{1z}}{k_z}} = \frac{k_z - k_{1z}}{k_z + k_{1z}}, \text{ where } k_\rho^2 + k_z^2 = k^2 = \omega^2\mu\epsilon \text{ and } k_\rho^2 + k_{1z}^2 = k_1^2 = \omega^2\mu\epsilon_1$$

by Eq. (3.4.25) on p. 379. The magnetic field for the VMD on a half-space is thus

$$H_z = \int_{-\infty}^{\infty} dk_\rho \left[-i\frac{IA\,k_\rho^3}{8\pi k_z}(1+R^{TE})\right] H_0^{(1)}(k_\rho\rho)\, e^{ik_z z}$$

Now, we want to find a closed form approximation to this integral as the observation point becomes far from the dipole. We will use the saddle-point method.

First, we use the large-argument approximation to $H_0^{(1)}$,

$$H_0^{(1)}(x) \sim \sqrt{\frac{2}{\pi x}}\, e^{ix - i\pi/4} = \sqrt{\frac{-2i}{\pi x}}\, e^{ix}, \quad x \to \infty$$

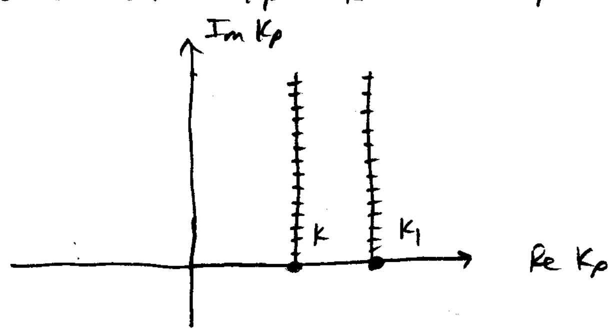
to simplify the integral to

$$H_z = \int_{-\infty}^{\infty} dk_\rho \left[-i\frac{IA\, k_\rho^3}{8\pi k_z}\left(1+R^{TE}\right)\right] \sqrt{\frac{-2i}{\pi k_\rho \rho}}\, e^{ik_\rho \rho}\, e^{ik_z z} \quad (z>0)$$

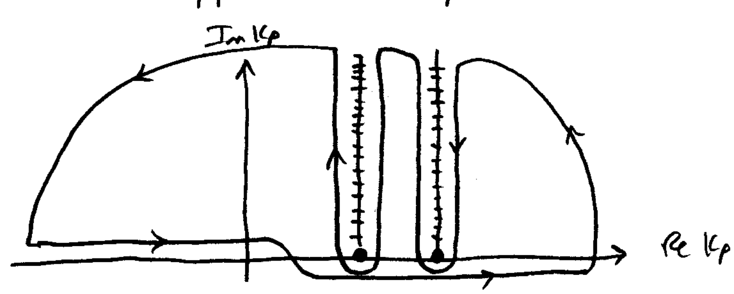
$$= -\frac{iIA}{8\pi}\sqrt{\frac{-2i}{\pi\rho}} \int_{-\infty}^{\infty} dk_\rho \frac{k_\rho^3}{k_z\sqrt{k_\rho}}\left(1+R^{TE}\right) e^{ik_\rho\rho + ik_z z}$$

$$= -\frac{iIA}{8\pi}\sqrt{\frac{-2i}{\pi\rho}} \int_{-\infty}^{\infty} \frac{k_\rho^3}{k_z\sqrt{k_\rho}}\, \frac{2k_z}{k_z+k_{1z}}\, e^{ik_\rho\rho + ik_z z}$$

Since $k_z = \pm\sqrt{k^2 - k_\rho^2}$ and $k_{1z} = \pm\sqrt{k_1^2 - k_\rho^2}$ are multivalued, there are branch points at $k_\rho = k$ and $k_\rho = k_1$:



The integration path is along the real axis. To evaluate the integral using complex contour integration, we move the path ϵ below the branch points at k and k_1 and close it in the upper half-plane:



Since we can't cross the branch cuts, the contour has to come down around each of the cuts. This is known as the Sommerfeld integration path (SIP).

There is a saddle point which is found from the zero of the derivative of the exponent:

$$\frac{d}{dk_\rho}\left(ik_\rho \rho + i\sqrt{k^2-k_\rho^2}\, z\right) = 0$$

$$i\rho + \frac{i\frac{1}{2}(-2k_\rho)}{\sqrt{k^2-k_\rho^2}}\, z = 0$$

$$r\sin A - \frac{k_\rho}{\sqrt{k^2-k_\rho^2}}\, r\cos A = 0$$

$$\sqrt{k^2-k_\rho^2}\,\sin A = k_\rho \cos A$$

$$(k^2-k_\rho^2)\sin^2 A = k_\rho^2 \cos A$$

$$k_\rho^2(\sin^2 A + \cos^2 A) = k^2 \sin^2 A$$

$$\underline{\underline{k_\rho = k\sin A}} \quad \rightarrow \quad \text{Saddle point}$$

Now, we need to find the steepest descent path. This is the path through the saddle point where the imaginary part of the exponent is equal to the value at the saddle point:

$$\mathrm{Im}\left\{ik_\rho \rho + i\sqrt{k^2-k_\rho^2}\, z\right\}\Big|_{k_\rho = k\sin A}$$

$$= \mathrm{Im}\left\{ik\sin A\,\rho + i\sqrt{k^2-k^2\sin^2 A}\, z\right\}$$

$$= \mathrm{Im}\left\{ikr\sin^2 A + i\,k\cos A\, r\cos A\right\}$$

$$= \mathrm{Im}\left\{ikr(\sin^2 A + \cos^2 A)\right\}$$

$$= kr$$

So that

$$\mathrm{Im}\left\{ik_\rho \rho + i\sqrt{k^2-k_\rho^2}\, z\right\} = kr$$

$$= \mathrm{Im}\left\{i(k_{\rho r} + ik_{\rho i})\, r\sin A + i\sqrt{k^2-(k_{\rho r}+ik_{\rho i})^2}\, r\cos A\right\}$$

$$= \mathrm{Im}\left\{i(k_{\rho R} + ik_{\rho I})\sin A + i\sqrt{k^2-(k_{\rho r}+ik_{\rho i})^2}\, \cos A\right\} r$$

Let $k_\rho = k\sin A + \alpha$. The imaginary part is

$$\mathrm{Im}\left\{i(k\sin A + \alpha)\sin A + i\sqrt{k^2-(k\sin A+\alpha)^2}\, \cos A\right\} r$$

$$= \mathrm{Im}\left\{ ik\sin^2 A + i\alpha \sin A + i\sqrt{k^2\cos^2 A - 2k\alpha\sin A - \alpha^2}\;\cos A \right\} r$$

$$= \mathrm{Im}\left\{ ik\sin^2 A + i\alpha \sin A + ik\cos^2 A\sqrt{1 - \frac{(2k\alpha\sin A + \alpha^2)}{k^2\cos^2 A}} \right\} r$$

$$\simeq \mathrm{Im}\left\{ ik\sin^2 A + i\alpha \sin A + ik\cos^2 A\left(1 - \frac{2k\alpha\sin A + \alpha^2}{2(k^2\cos^2 A)} - \frac{1}{8}\left(\frac{2\alpha k\sin A + \alpha^2}{k^2\cos^2 A}\right)^2\right) \right\} r$$

$$\cong \mathrm{Im}\left\{ ik\sin^2 A + i\alpha \sin A + ik\cos^2 A - i\,\frac{2k\alpha\sin A + \alpha^2}{2k} - \frac{i\alpha^2\sin^2 A}{k\cos^2 A} \right\} r$$

$$= \mathrm{Im}\left\{ ik + i\alpha\sin A - i\alpha\sin A - \frac{i\alpha^2}{2k}\left(1 + \frac{\sin^2 A}{\cos^2 A}\right) \right\} r$$

$$= \mathrm{Im}\left\{ ik - \frac{i\alpha^2}{2k}\frac{1}{\cos^2 A} \right\}$$

$$= \mathrm{Im}\left\{ ikr - \frac{i(\alpha_R + i\alpha_I)^2 r}{2k\cos^2\theta} \right\}$$

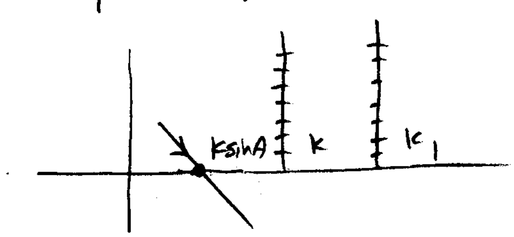
$$= \mathrm{Im}\left\{ ikr - i\,\frac{(\alpha_R^2 - \alpha_I^2) r}{2k\cos^2\theta} + \frac{\alpha_R\alpha_I r}{k\cos^2 A} \right\}$$

$$= kr - \frac{(\alpha_R^2 - \alpha_I^2) r}{2k\cos^2 A}$$

For this to be equal to kr, we must have

$$\alpha_R = \pm \alpha_I$$

We choose the negative sign, because the exponent blows up along the path $\alpha_R = \alpha_I$. The SDP is



We now evaluate the integral along the SDP:

$$H_z \approx -\frac{iIA}{8\pi}\sqrt{\frac{-2i}{\pi\rho}}\;\frac{k^3\sin^3 A}{k_z\sqrt{k\sin A}}\;\frac{2k_z}{k\cos A + \sqrt{k_1^2 - k\sin^2 A}}\int ds\,\frac{d\alpha}{ds}\,e^{ikr - s^2 r/k\cos^2\theta}$$

$$= -\frac{iIA}{4\pi}\sqrt{\frac{-2i}{\pi r}}\;\frac{k^2\sin^2 A}{\cos A + \sqrt{k_1^2/k^2 - \sin^2 A}}\;e^{ikr}\int ds\,\frac{d\alpha}{ds}\,e^{-(r/k\cos^2\theta)s^2}$$

Now, $\dfrac{d\alpha}{ds} = \dfrac{d(s - is)}{ds} = (1 - i) = \sqrt{-2i}$

So that

$$H_z \simeq -\frac{iIA(-2i)}{4\pi\sqrt{\pi r k}}\frac{k^2\sin^2A}{\cos\theta+\sqrt{k_1^2/k^2-\sin^2A}}\,e^{ikr}\int ds\, e^{-(r/k\cos^2A)s^2}$$

$$= -\frac{IA\,e^{ikr}}{2\pi\sqrt{\pi r k}}\frac{k^2\sin^2A}{\cos A+\sqrt{k_1^2/k^2-\sin^2A}}\sqrt{\frac{\pi}{(r/k\cos^2A)}}$$

$$= -\frac{IA\,e^{ikr}}{2\pi r}\frac{k^2\sin^2A\cos A}{\cos A+\sqrt{k_1^2/k^2-\sin^2A}}$$

To check this answer, let's set $k_1 = k$. The field should reduce to the far field of a VMD in free space

$$H_z = -\frac{IA\,e^{ikr}}{2\pi r}\frac{k^2\sin^2A\cos\theta}{\cos A+\sqrt{1-\sin^2A}}$$

$$= -\frac{IA\,e^{ikr}}{2\pi r}\frac{k^2\sin^2A\cos A}{2\cos\theta}$$

$$= -\frac{IA\,e^{ikr}}{4\pi r}k^2\sin^2\theta \qquad \text{(VMD in free space)}$$

which is the correct field (by comparison to (4.3.17) on p 520.).