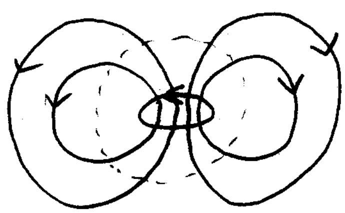
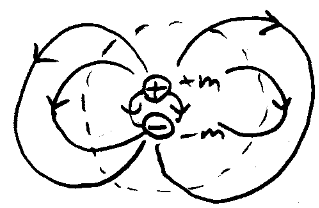
Many important theorems and methods, such as duality, the equivalence principle, the image theorem, and so forth, are based on the foundation of the uniqueness theorem.

The uniqueness theorem is that if the tangential electric or magnetic field is specified over the entire boundary of a region, and the sources inside are fixed, then the field solution within the region is uniquely determined.

One important class of applications of the uniqueness theorem are various types of equivalence principles. For example, the magnetic fields on a small surface around a current loop are equal to the magnetic field on the same surface around a magnetic dipole:



current loop



Magnetic dipole

Since the fields on the dotted surface in the two pictures are equal, by the uniqueness theorem the fields outside are equal. Thus, we say the current loop and magnetic dipole are equivalent. Inside the surface, near the sources, however, the fields are different.

<u>Proof of uniqueness theorem</u>:

We now prove the uniqueness theorem. Suppose that inside a surface S on which E_t or H_t is specified, there are two possible solutions to Maxwell's equations: $\vec{E}_1, \vec{H}_1$ and $\vec{E}_2, \vec{H}_2$. By linearity, the difference fields

$$\delta\vec{E} = \vec{E}_1 - \vec{E}_2$$
$$\delta\vec{H} = \vec{H}_1 - \vec{H}_2$$

satisfy Maxwell's equations,

$$\nabla\times\delta\vec{E} = i\omega\mu\,\delta\vec{H}$$

$$\nabla\times\delta\vec{H} = -i\omega\epsilon\,\delta\vec{E}$$

We dot Faraday's law with $\delta\vec{H}^*$ and the conjugate of Ampere's law with $\delta\vec{E}$:

$$\delta\vec{H}^* \cdot \nabla\times\delta\vec{E} = i\omega\mu\, \delta\vec{H}^* \cdot \delta\vec{H}$$

$$\delta\vec{E} \cdot \nabla\times\delta\vec{H}^* = -i\omega\epsilon^*\, \delta\vec{E} \cdot \delta\vec{E}^*$$

Subtracting,

$$\delta\vec{H}^* \cdot \nabla\times\delta\vec{E} - \delta\vec{E}\cdot\nabla\times\delta\vec{H}^* = i\omega\mu|\delta\vec{H}|^2 - i\omega\epsilon^*|\delta\vec{E}|^2$$

Using the identity $\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot\nabla\times\vec{A} - \vec{A}\cdot\nabla\times\vec{B}$, we have

$$\nabla\cdot(\delta\vec{E}\times\delta\vec{H}^*) = i\omega\mu|\delta\vec{H}|^2 - i\omega\epsilon^*|\delta\vec{E}|^2$$

Taking the complex conjugate,

$$\nabla\cdot(\delta\vec{E}^*\times\delta\vec{H}) = -i\omega\mu^*|\delta\vec{H}|^2 + i\omega\epsilon|\delta\vec{E}|^2$$

Adding these two equations and integrating over the volume V inside S,

$$\int_V dv\, \nabla\cdot(\delta\vec{E}\times\delta\vec{H}^* + \delta\vec{E}^*\times\delta\vec{H}) = \omega\int_V dv\left[\underbrace{(i\mu - i\mu^*)}_{-2\mu_I}|\delta H|^2 + \underbrace{(-i\epsilon^* + i\epsilon)}_{-2\epsilon_I}|\delta E|^2\right]$$

Using the divergence theorem, we find

$$\oint_S d\vec{S}\,(\delta\vec{E}\times\delta H^* + \delta\vec{E}^*\times\delta\vec{H}^*) = -2\omega\int_V dv\left(\mu_I|\delta\vec{H}|^2 + \epsilon_I|\delta\vec{E}|^2\right)$$

The left hand side of this equation must be zero, since the tangential components of $\vec{E}$ and $\vec{H}$ are by hypothesis specified on S. Thus,

$$\int_V \left(\mu_I|\delta\vec{H}|^2 + \epsilon_I|\delta\vec{E}|^2\right) dv = 0$$

Since both terms are positive (note that $\mu_I > 0$ and $\epsilon_I > 0$, unless the medium is amplifying rather than lossy), we must have

$$\delta\vec{H} = \delta\vec{E} = 0$$

on the entire region V, and the theorem is done.

There are a few interesting questions about this theorem:

① Maxwell's equations are 2^{nd} order. Why, then, did we only need to specify $\vec{E}_+$ <u>or</u> $\vec{H}_+$ on S?

② If the medium is lossless, so that $\epsilon_I = \mu_I = 0$, the proof breaks down. Why?

③ How is Huygens' principle (i.e., the radiation integral for $\vec{E}$ in terms of $\vec{J}$, $\vec{M}$, and tangential fields on a surface) related to the uniqueness theorem?

The duality theorem results from interchanging electric and magnetic sources. If we make the substitutions

$$\vec{E} \to \vec{H}$$

$$\vec{H} \to -\vec{E}$$

$$\mu \to \epsilon$$

$$\epsilon \to \mu$$

$$\vec{J} \to \vec{M}$$

$$\vec{M} \to -\vec{J}$$

Then Maxwell's equations,

$$\nabla \times \vec{E} = +i\omega\mu \vec{H} - \vec{M}$$

$$\nabla \times \vec{H} = i\omega\epsilon \vec{E} + \vec{J}$$

become

$$\nabla \times \vec{H} = -i\omega\epsilon \vec{E} + \vec{J}$$

$$\nabla \times \vec{E} = i\omega\mu \vec{H} + \vec{M}$$

which is identical to the original equations!

Thus, if we have an electric source, for example, then we can find the fields due to an equal magnetic source by swapping $\vec{E}$ and $\vec{H}$ and ϵ and μ.

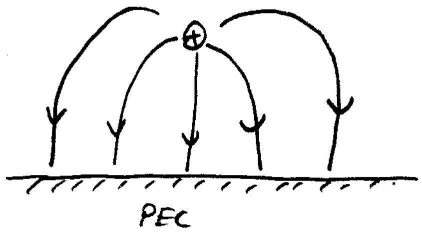
For example, a Hertzian dipole with $\vec{J} = \hat{z} I\ell\, \delta(\vec{r})$ A/m^2 has

$$\vec{E}_{far} = -\frac{i\omega\mu I\ell e^{ikr}}{4\pi r} \sin\theta\, \hat{\theta}$$

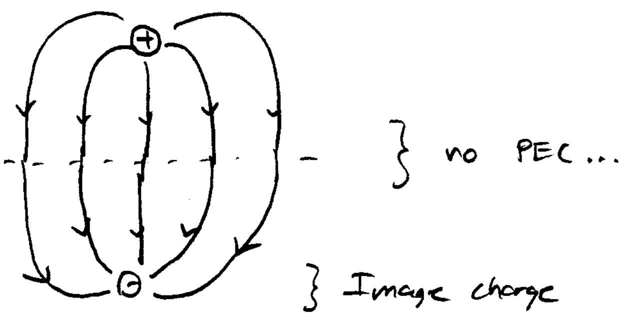
For a magnetic dipole with $\vec{M} = \hat{z} I\ell\, \delta(\vec{r})$ wb/s /m^2,

$$\vec{H}_{far} = \frac{-i\omega\epsilon I\ell e^{ikr}}{4\pi r} \sin\theta\, \hat{\theta} .$$

Using the uniqueness theorem, we can transform a problem with a source over a PEC plane to an equivalent source in free space: Consider a charge over a PEC plane:

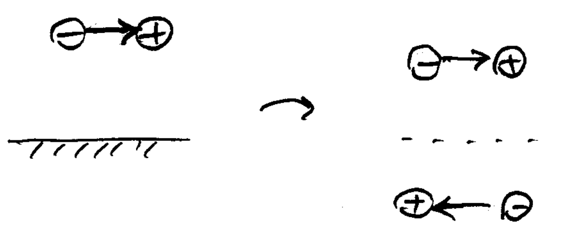


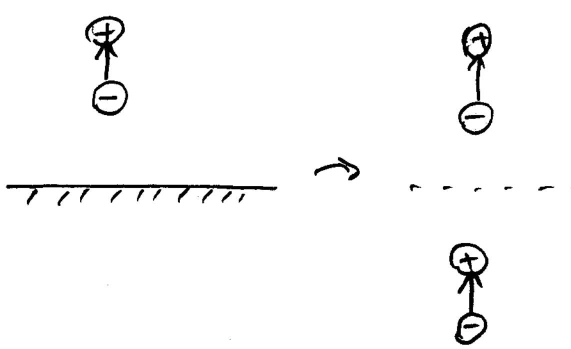
The electric field is perpendicular to the plane, and $E_{tan} = 0$ along the plane. If we add another opposite charge an equal distance below the plane, the field at the boundary is unchanged:



By the uniqueness theorem, the fields above the boundary do not change. Thus, we have an equivalent problem, where the fields in the region of interest (above the PEC) are unchanged.

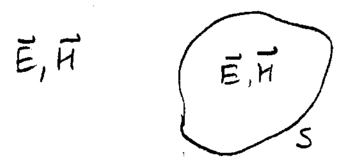
We can find equivalents to dipoles:





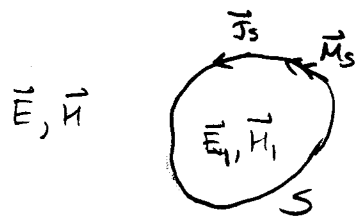
The equivalence principle is a consequence of the uniqueness theorem, and can be used for a wide variety of EM problems.

Consider electromagnetic fields $\vec{E}$ and $\vec{H}$ in a region of interest outside a surface S. The fields $\vec{E}$ and $\vec{H}$ are defined both inside and outside of S:



(original problem)

Now suppose we want to change the fields outside the region of interest to different fields $\vec{E}_1$ and $\vec{H}_1$:



(Equivalent problem)

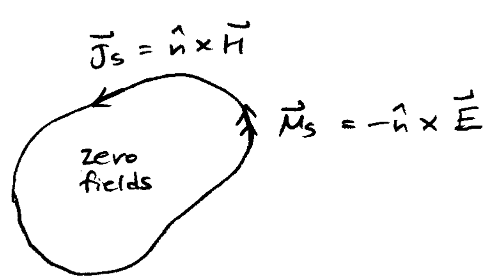
In order for this new "equivalent problem" to satisfy Maxwell's equations, we must place surface currents on the surface S as given by the electromagnetic boundary conditions:

$$\vec{J}_s = \hat{n} \times (\vec{H} - \vec{H}_1)$$

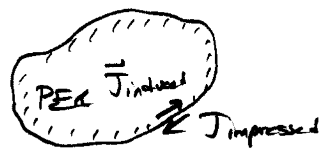
$$\vec{M}_s = -\hat{n} \times (\vec{E} - \vec{E}_1)$$

The key idea here is that the fields in the region of interest (outside of S) have not changed, but the fields and sources elsewhere have changed. We say that the two problems are <u>equivalent</u>.

Since E_1 and H_1 are arbitrary, we can get several variants of this surface equivalence principle. The first is obtained by choosing $\vec{E}_1 = \vec{H}_1 = 0$:

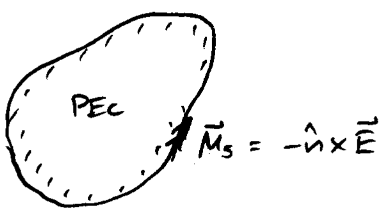


This is known as Love's equivalence principle. Next, we fill the inside of S with PEC — which we can do without changing the fields outside, since the fields inside are zero. We now use the following fact: an electric current impressed at the surface of a PEC induces an equal and opposite current on the PEC, and the total radiated field due to the electric current is zero:



$$\vec{E} = \vec{H} = 0$$

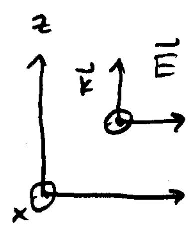
(The impressed electric current is 'shorted' by the PEC.) Thus, we can set $\vec{J}_s = 0$ without changing the fields outside S:



$$\vec{E}, \vec{H}$$

To better understand the surface equivalence principle, let's look at an example:

<u>Plane wave in free space</u>



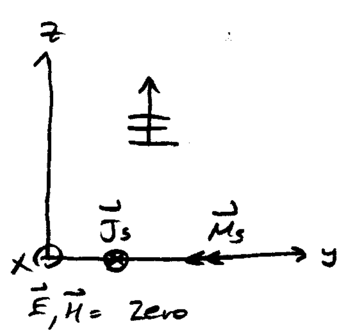
$$\vec{E} = \hat{x} E_0 e^{ikz}$$
$$\vec{H} = \hat{y} \frac{E_0}{\eta} e^{ikz}$$

Let the region of interest be $z > 0$. We have many possible equivalent problems:

① $\vec{J}_s = \hat{n}\times\vec{H} = \hat{z}\times\hat{y}\frac{E_0}{\eta}e^{ikz}\big|_{z=0} = -\hat{x}E_0/\eta$ (Love's equivalent)

$\vec{M}_s = -\hat{n}\times\vec{E} = -\hat{z}\times\hat{x}E_0 e^{ikz}\big|_{z=0} = -\hat{y}E_0$

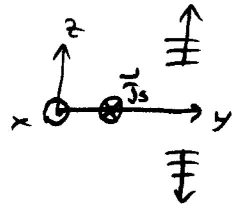
The fields are zero for $z < 0$.



② Let $\vec{E} = \hat{x} E_0 e^{-ikz}$, $\vec{H} = -\hat{y} \frac{E_0}{\eta} e^{-ikz}$ for $z < 0$

Then $\vec{J}_s = \hat{z} \times \left(\hat{y} \frac{E_0}{\eta} - \left(-\hat{y} \frac{E_0}{\eta}\right)\right) = -2\hat{x} \frac{E_0}{\eta}$

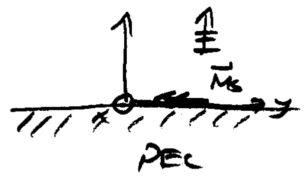
$\vec{M}_s = -\hat{z} \times \left(\hat{x} E_0 - \hat{x} E_0 \right) = 0$



③ Fill $z<0$ with PEC. Then

$J_s = 0$

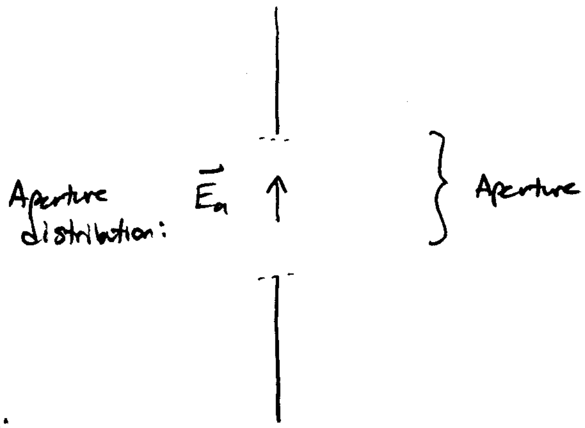
$M_s = -\hat{n} \times \vec{E} = -\hat{y} E_0$



There are other possibilities as well...

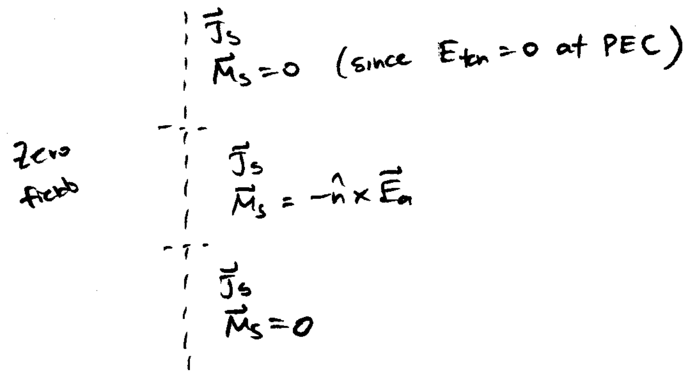
Another example is the aperture antenna:

<u>Problem</u>: Find a $\vec{J}_s$ and/or $\vec{M}_s$ that radiate the same fields as an aperture antenna:

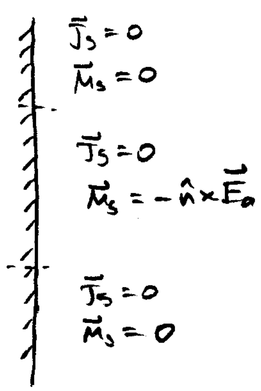


<u>Solution</u>:

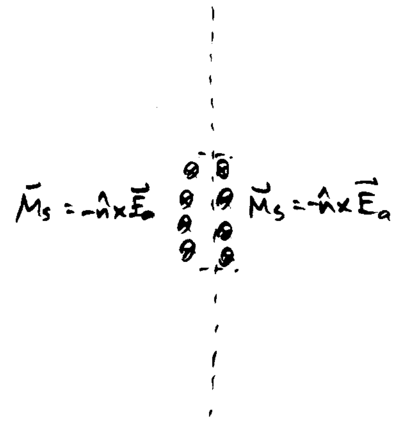
A. Love's equivalent:



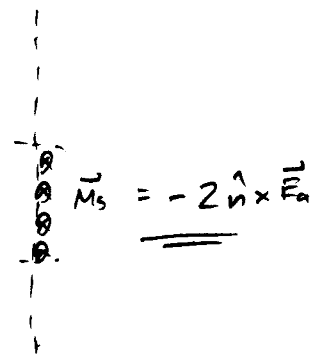
B. Fill with PEC to short out $\vec{J}_s$:



C. Use image theorem:



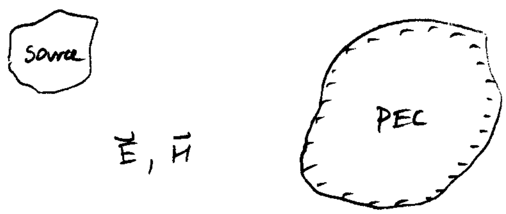
D. Combine image and original source:



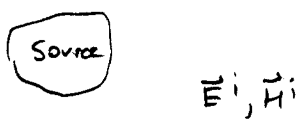
Now we have a current in free space that exists only over the aperture, and we can use the free space radiation integral to find the radiated fields!

PHYSICAL EQUIVALENT

Another reformulation of the equivalence principle that is useful in scattering problems is the physical equivalent. Let's assume that a source produces a field that strikes an object:



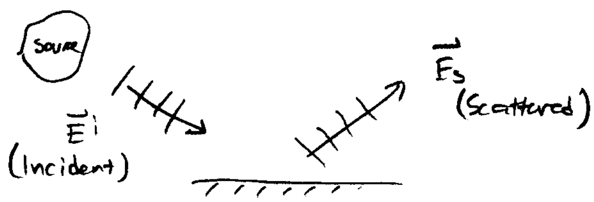
The resulting field is $\vec{E}, \vec{H}$. Now define $\vec{E}^i$ and $\vec{H}^i$ to be the fields radiated if the source were in free space:



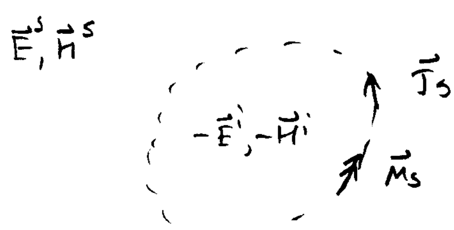
This is the incident field that illuminates the object. Now, we define a scattered field, such that

$$\vec{E} = \vec{E}^i + \vec{E}^s$$

$$\vec{H} = \vec{H}^i + \vec{H}^s$$



Now, we want to find an equivalent current on the surface of the object that radiates $\vec{E}^s, \vec{H}^s$ in free space:



Using the idea of the surface equivalence principle, we find that

$$\vec{J}_s = \hat{n} \times (\vec{H}^s - (-\vec{H}^i)) = \hat{n} \times \vec{H}$$

$$\vec{M}_s = -\hat{n} \times (\vec{E}^s - (-\vec{E}^i)) = \hat{n} \times \vec{E} = 0 \quad \left(\text{Since } E_{tan} = 0 \text{ at a PEC surface.} \right)$$

Note that if we do not know the solution $\vec{E}, \vec{H}$ to the original problem, we do not know $\vec{J}_s = \hat{n} \times \vec{H}$.

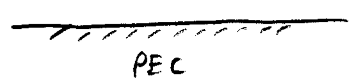
But we can still use this result in several ways:

Physical optics approximation

If the PEC surface is an infinite plane, then we can find H^s and E^s:

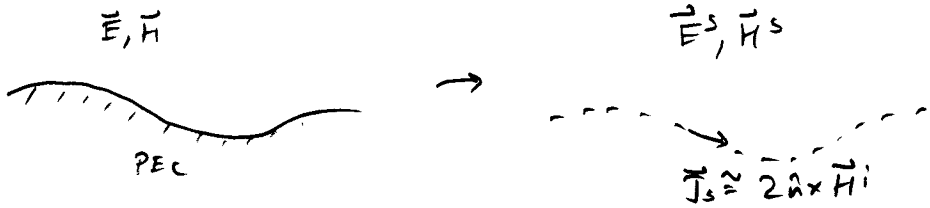
$$E^s = -E^i$$

$$H^s = H^i$$



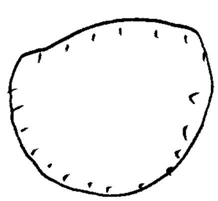
using boundary conditions. Thus, $\vec{J}_s = 2\hat{n} \times \vec{H}^i$, which is known.

If the PEC surface is no longer flat, $\vec{J}_s \neq 2\hat{n} \times \vec{H}^i$, but we can use this as an approximation:



Integral Equation

We can reformulate Maxwell's equations as an integral equation using the physical equivalent:



$$E_{tan} = 0 \rightarrow \hat{n} \times (\vec{E}^i + \vec{E}^s) = 0$$

Using the radiation integral,

$$\underset{\substack{\Uparrow \\ \text{known}}}{\hat{n} \times \vec{E}^i} = -\hat{n} \times i\omega\mu \left[\bar{\bar{I}} + \frac{\nabla\nabla}{k^2} \right] \int d\vec{r}\,' \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|} \underset{\substack{\Uparrow \\ \text{unknown}}}{\vec{J}_s(\vec{r}\,')}$$

which is an integral equation, and can be solved for the unknown surface current.

When do we use the various equivalence principles? To simplify a problem, by:

① Going from a source and scatterer or aperture to a source in free space. (The equivalent problem may still be difficult if the equivalent source is in terms of the original unknown fields. It may be possible to approximate the equivalent source...)

② Going from a source of infinite extent to an equivalent source of finite extent (e.g., the aperture problem).