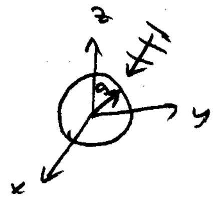
Rayleigh Scattering

Let us first consider the case of a dielectric sphere with a small radius, such that

$$a << \lambda$$



Let the incident field be $\bar{E}^i = \hat{z} E_0 e^{ikx}$. For a small sphere, the electric field induces a dipole moment, and the sphere re-radiates a scattered field as if it were a dipole antenna.

Since the radius is small, we can approximate the dipole fields by the $1/r^3$ terms:

$$\bar{E}_s = \frac{ik\eta I\ell}{4\pi r} \frac{1}{(kr)^2} (\hat{r}\, 2\cos\theta + \hat{\theta} \sin\theta) + O(1/r^2)$$

$$\bar{H}_s = O(1/r^2)$$

(Scattered field just outside the sphere)

Inside the sphere,

$$\vec{E} = \hat{z} E_i = (\hat{r} \cos\theta - \hat{\theta} \sin\theta) E_i$$

At the surface $r = a$, we have the boundary conditions

$$\hat{n} \times (\bar{E}_1 - \bar{E}_2) = 0$$

$$\hat{n} \cdot (\bar{D}_1 - \bar{D}_2) = 0$$

All we need to do is to use these boundary conditions to find $I\ell$. For convenience, let's set

$$E_s = \frac{i\eta}{4\pi k a^3} I\ell$$

so that

$$E_s = (\hat{r}\, 2\cos\theta + \hat{\theta} \sin\theta)(a/r)^3 E_s$$

Outside the sphere, $\bar{E}^i = \hat{z} E_0 e^{ikx} \simeq (\hat{r}\cos\theta - \hat{\theta}\sin\theta) E_0$. From the $\bar{E}$ boundary condition,

$$E_{1\theta} = E_{2\theta} \Rightarrow -E_0 \sin\theta + E_s \sin\theta = -E_i \sin\theta$$

$$\Rightarrow -E_0 + E_s = -E_i$$

From the $\bar{D}$ boundary condition,

$$D_{1r} = D_{2r} \Rightarrow \epsilon_0 E_0 \cos A + 2\epsilon_0 \cos A\, E_s = \epsilon_1 E_i \cos A$$

$$\Rightarrow \epsilon_0 E_0 + 2\epsilon_0 E_s = \epsilon_1 E_i$$

Solving for E_s,

$$E_s = \frac{\epsilon_1 - \epsilon_0}{\epsilon_1 + 2\epsilon_0} E_0$$

So, we have

$$I\ell = \frac{4\pi k a^3}{i\eta} \frac{\epsilon_1 - \epsilon_0}{\epsilon_1 + 2\epsilon_0} E_0$$

Far away from the sphere, the scattered field is

$$E_\theta^s = \frac{-ik\eta e^{ikr}}{4\pi r} I\ell \sin A$$

$$= -\frac{\cancel{i}k\cancel{\eta} e^{ikr}}{\cancel{4\pi} r} \frac{\cancel{4\pi} k a^3}{\cancel{i\eta}} \frac{\epsilon_1 - \epsilon_0}{\epsilon_1 + 2\epsilon_0} E_0 \sin A$$

$$= -\frac{e^{ikr}}{r} k^2 a^3 E_0 \sin A \left(\frac{\epsilon_1 - \epsilon_0}{\epsilon_1 + 2\epsilon_0}\right)$$

The scattered power density is

$$S_r^s = \frac{|E^s|^2}{2\eta} = \frac{1}{2\eta} \frac{1}{r^2} \underbrace{k^4}_{*} a^6 |E_0|^2 \sin^2 A \left(\frac{\epsilon_1 - \epsilon_0}{\epsilon_1 + 2\epsilon_0}\right)^2$$

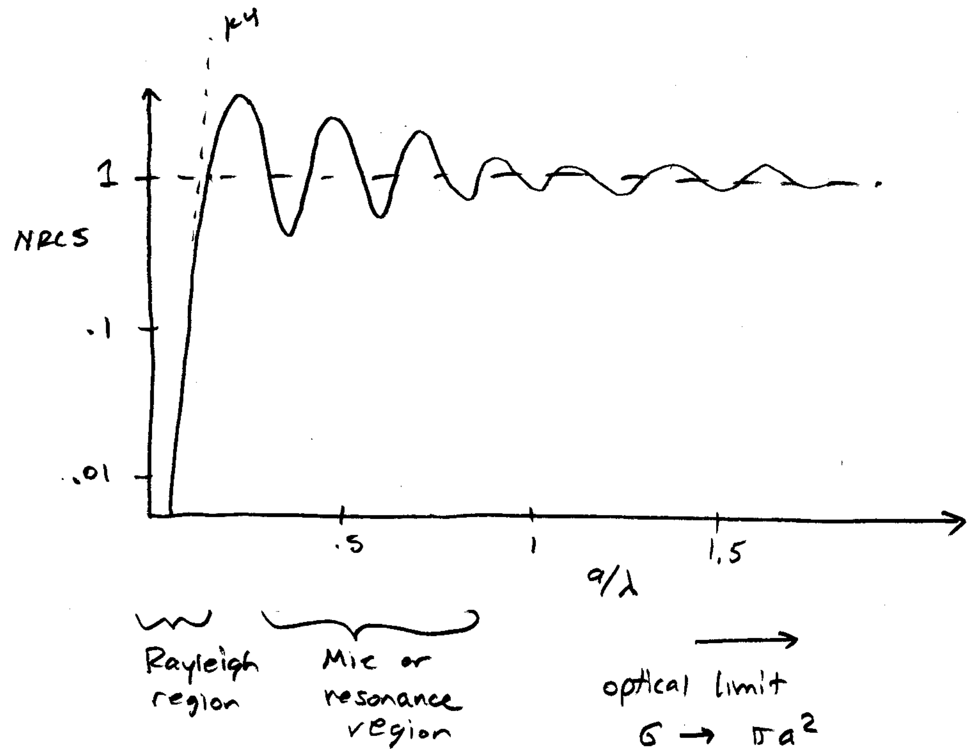
Notice that the scattered power scales as the fourth power of the wavenumber.

of course, the k^4 dependance can't continue indefinately, because eventually the wavelength is comparable to the sphere radius.

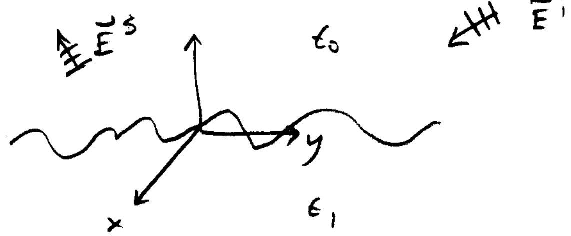
If we look at the normalized monostatic RCS,

$$NRCS = \frac{\sigma_{3D}}{\pi a^2}$$

we find that



A rough surface is a medium which is planar with a perturbation:



so that the profile of the interface is $z = f(x,y)$. We can study the problem of rough surface scattering in two ways:

Deterministic: Find scattered field for a single surface $z = f(x,y)$.

Stochastic: Find statistics of scattered field in terms of statistics of the surface.

The second approach is far more important than the first, because we generally make many measurements of scattered fields over different pieces of a surface, or we do not even have the deterministic profile, as with the ocean surface, which is always changing.

How do we describe a rough surface statistically? Generally in terms of a height probability distribution

$$p(z) = \text{surface height pdf}$$

and a correlation function:

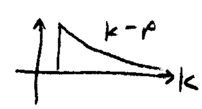
$$R(\vec{r}) = \int f(\vec{r}\,')\, f(\vec{r}\,' + \vec{r})\, d\vec{r}\,' \quad , \quad \vec{r} = x\hat{x} + y\hat{y}.$$

The surface height pdf is usually Gaussian, and its variance is h^2, where h is the surface height standard deviation. Common correlation functions are

Exponential: $e^{-|\vec{r}|/L}$

Gaussian: $e^{-|\vec{r}|^2/2L^2}$

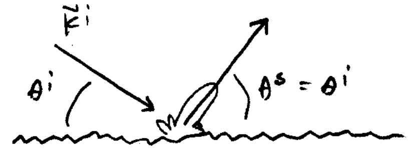
Power law: $F\{R(\vec{r})\} = C|\vec{k}|^{-p}$



Some surfaces vary only in one direction, so that the EM scattering problem becomes a 2D problem. We refer to these as 1D surfaces.

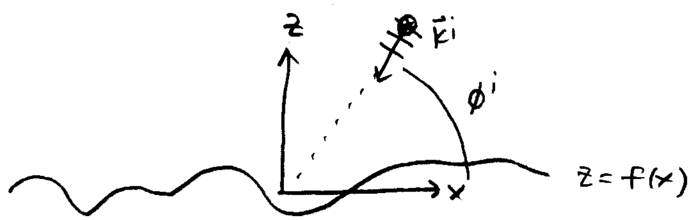
The Fourier transform of R is the surface height power spectral density, which gives the power in sinusoidal surface components in terms of the spatial frequency, and is given the notation $S(\vec{k})$ or $W(\vec{k})$.

We can make a few qualitative observations about rough surface scattering. If the surface height standard deviation is small, then most of the energy will be scattered in the "specular" direction:



The scattered field will have a strong coherent component. On the other hand, for large h, the scattered field will be diffuse:

We will study the case of a 1D rough surface using the physical optics approximation. Let the profile be $z = f(x)$, so that y is the invariant direction. This becomes a 2D EM problem.



There are two polarizations, TM^y and TE^y. Let's choose TM^y, so that the incident E field is in the y direction. If the surface were a flat PEC plane, then the current flowing on the surface is known:

$$\vec{J}_s = 2\hat{n} \times \vec{H}^i \quad (\text{flat case})$$

Since the surface is not flat, this becomes an approximation:

$$\vec{J}_s \simeq 2\hat{n} \times H^i \Big|_{z=f(x)}$$

We can think of this current as a collection of line currents, and get the scattered field by integrating the field due to each line current:

$$\vec{E} = -\frac{\omega\mu}{4}\hat{y}\int_S H_0^{(1)}(k|\vec{r}-\vec{r}'|)\, J_s(\vec{r}')$$

We need to find the amplitude of $\vec{J}_s$:

$$\vec{J}_s = 2\hat{n} \times \frac{E_0}{\eta}\left[\hat{x}\sin\phi^i - \hat{z}\cos\phi^i\right] e^{-ik(x\cos\phi^i + z\sin\phi^i)} \Big|_{z=f(x)}$$

The surface normal can be found from

$$\hat{n} = \frac{\nabla(z - f(x))}{\|\nabla(z - f(x))\|}$$

$$= \frac{\hat{z} - \hat{x}\frac{\partial f}{\partial x}}{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2}}$$

So that

$$\vec{J}_s = \frac{2E_0}{\eta}\frac{1}{\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2}}\left(\hat{z}-\hat{x}\frac{\partial f}{\partial x}\right)\times\left(\hat{x}\sin\phi^i-\hat{z}\cos\phi^i\right)e^{+i\vec{k}^i\cdot\vec{r}}$$

$$= \frac{2E_0}{\eta}\frac{1}{\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2}}\left(\hat{y}\sin\phi^i-\hat{y}\frac{\partial f}{\partial x}\cos\phi^i\right)e^{i\vec{k}^i\cdot\vec{r}}$$

$$= \hat{y}\underbrace{\frac{2E_0}{\eta}\frac{1}{\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2}}\left(\sin\phi^i-\frac{\partial f}{\partial x}\cos\phi^i\right)e^{i\vec{k}^i\cdot\vec{r}}}_{J_s}$$

The scattered far field is

$$\vec{E} \simeq -\frac{\omega\mu}{4}\hat{y}\int_s\sqrt{\frac{-2i}{\pi kr}}\,e^{+ikr}e^{-ik\hat{r}\cdot\vec{r}\,'}J_s(\vec{r}\,')\,d\vec{r}\,' \quad \text{if } \left(\frac{\partial f}{\partial x}\right)^2 \ll 1$$

$$= -\frac{\omega\mu}{4}\hat{y}\sqrt{\frac{-2i}{\pi kr}}\,e^{ikr}\int_s e^{-ik\hat{r}\cdot\vec{r}\,'}J_s(\vec{r}\,')\,d\vec{r}\,'$$

$$= \hat{y}\sqrt{\frac{-2i}{\pi kr}}\,e^{ikr}\left[-\frac{\omega\mu}{4}\cdot\frac{2}{\eta}\int_s\left(\sin\phi^i-\frac{\partial f}{\partial x}\cos\phi^i\right)e^{+i(\vec{k}^i-\vec{k}^s)\cdot\vec{r}\,'}d\vec{r}\,'\right]E_0$$

$$\simeq \hat{y}\sqrt{\frac{-2i}{\pi kr}}\,e^{ikr}\left[-\frac{k}{2}\sin\phi^i\int_s e^{i(\vec{k}^i-\vec{k}^s)\cdot\vec{r}\,'}d\vec{r}\,'\right] \quad \text{if } \frac{\partial f}{\partial x} \ll 1$$

What we have at this point is an approximation to the electric far field scattered by a PEC surface with a fixed profile $f(x)$.

The next step is to take this deterministic result and make it stochastic, so that instead of a field expression that depends on a specific profile, we have an expression that depends on the statistical properties of the surface.

We want to compute the time average of the field, assuming that the surface is changing (or moving) from instant to instant, but the statistical properties are time-invariant. We can either average the field or the power measurements:

Field: $\langle E \rangle$ → Only coherent component remains

Power: $\langle |E|^2 \rangle$ → Both coherent and diffuse components, so

that $P_{diffuse} = \langle |E|^2 \rangle - |\langle E \rangle|^2$

Often, only the diffuse scattering is of interest.

We want to find the time-averaged normalized scattering width,

$$\sigma_0 = \lim_{r \to \infty} \frac{1}{L} \left\langle \frac{2\pi\rho |E^s|^2}{|E_0|^2} \right\rangle$$

where L is the length of the surface illuminated by the incident wave. σ_0 can be thought of as the scattering width σ of each unit length piece of the surface.

If we plug in the previous expression for the scattered field, the time-average brackets can be taken inside the integrals over the surface. The z coordinate in the integrand is equal to the surface profile $f(x)$. If the averaging is then converted from time average to ensemble averaging, then the $f(x)$'s become correlation functions, and we have after a bit of work,

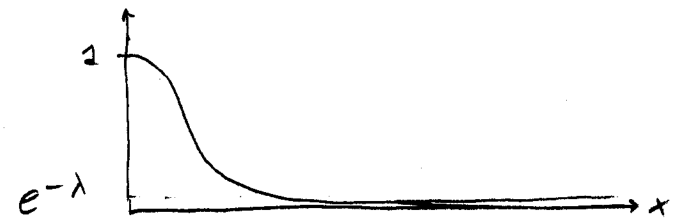
$$\sigma_0 = \frac{k}{\cos^2\theta} \int_{-L/2}^{L/2} dx\, e^{ik_b x} e^{-\lambda(1 - C(x))}$$

where $k_b = 2k\sin\theta$, and $\lambda = 4h^2k^2\cos^2\theta$. $C(x)$ is the normalized surface height correlation coefficient, and is defined by

$$C(x) = \frac{R(x)}{R(0)}$$

$$= h^{-2} R(x).$$

We can evaluate this integral asymptotically as $\lambda \to \infty$. The magnitude of the integrand looks like this:



Since $C(0) = 1$, the magnitude at $x = 0$ is 1. As $x \to \infty$, $C(x)$ goes to zero, so the magnitude of the integrand goes to $e^{-\lambda}$.

Since most of the contribution to the integral is near the origin, we can expand the integrand around $x = 0$:

$$G_0 \simeq \frac{k}{\cos^2\theta} \int dx \; e^{-\lambda(1-(1 - C''(0)/2\, x^2))} \, e^{ik_b x}$$

where the $C'(0)$ term drops out, due to the symmetry of $C(x)$. This leads to

$$G_0 = \frac{k}{\cos^2\theta} \int dx \; e^{-\lambda C''(0)/2\, x^2} \, e^{ik_b x}$$

Using the identity

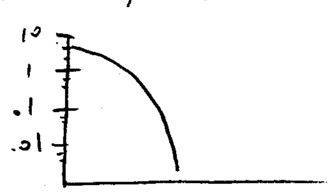
$$\int_{-\infty}^{\infty} dx \; e^{iax - bx^2} = \sqrt{\frac{\pi}{b}} \; e^{-a^2/4b}$$

we find that

$$G_0 = \frac{k}{\cos^2 A} \sqrt{\frac{\pi}{\lambda C''(0)/2}} \; e^{-k_b^2/4 \cdot \lambda C''(0)/2}$$

$$= \frac{k}{\cos^2 A} \sqrt{\frac{2\pi}{\lambda C''(0)}} \; e^{-k_b^2/2\lambda C''(0)}$$

The scattering width looks something like this:



This result is called the "geometrical optics" limit of the physical optics approximation.