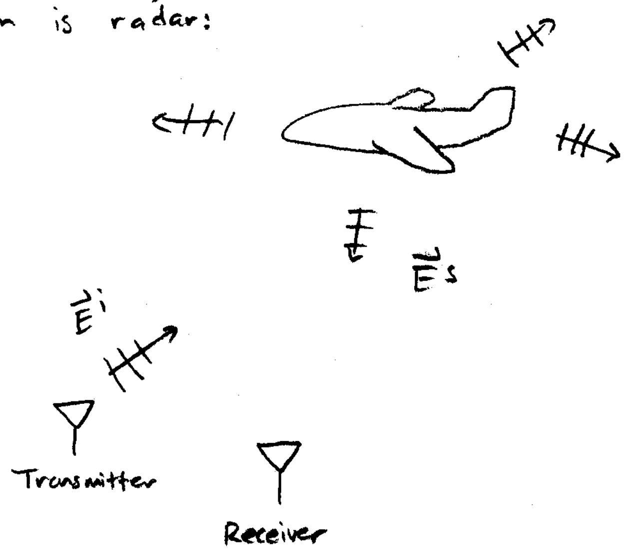
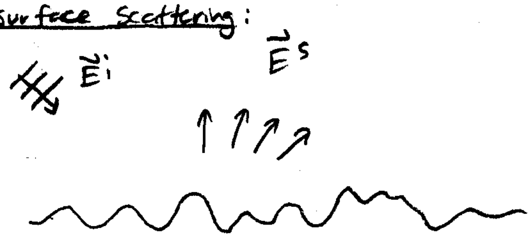
Scattering problems are EM BVP's for which the region of interest is exterior to a material object. Scattering problems occur in many applications, but the "canonical" scattering problem is radar:



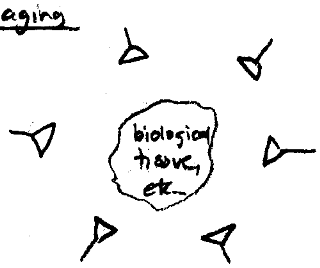
If the receiver and transmitter are in different locations, we refer to this as bistatic scattering. More commonly, the two antennas are in the same location, or the same antenna serves as both. This is monostatic scattering.

Other examples of scattering problems are:

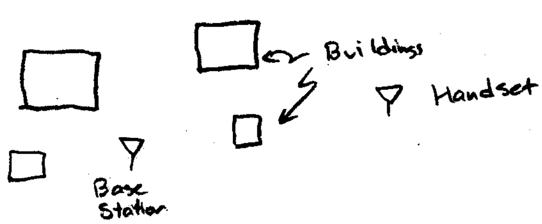
Rough surface scattering:



Imaging



Wireless propagation:

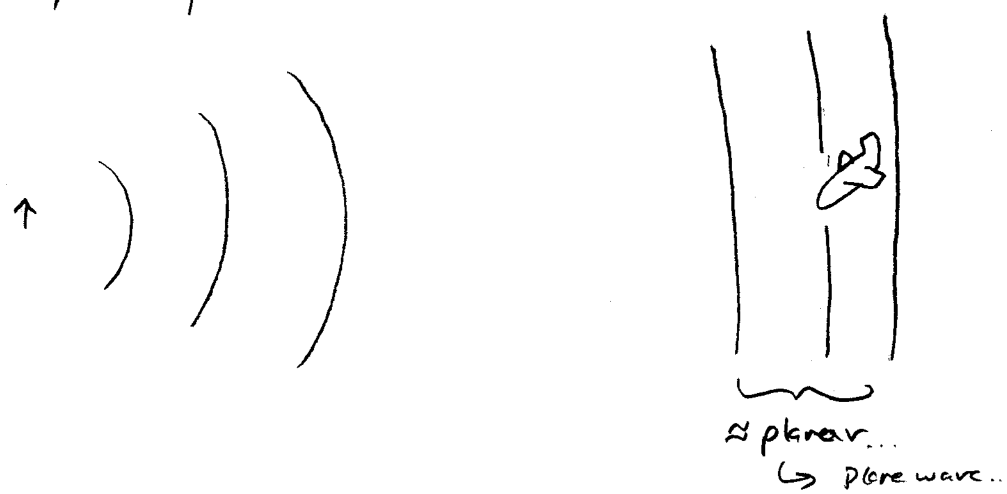


Incident field:

In general, the incident field is arbitrary, and is determined by the specific problem. The most common case, however, is a plane wave:

$$\vec{E}^i = \vec{E}_0^i e^{i\vec{k}^i \cdot \vec{r}}$$

The plane wave is important for two reasons. First, if the incident field is produced by the antenna, and if the antenna is in the far field of the scatterer, then the spherical wave produced by the antenna is approximately a plane wave at the scatterer:



Second, any incident field can be expressed as an integral over plane waves, so in principle plane wave scattering is all we need. (In practice, this idea is not used very often.)

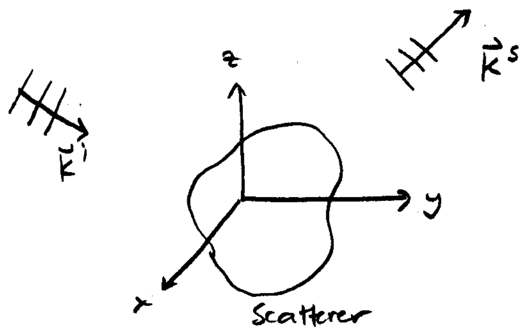
Scattered field:

For a given scattering problem, we generally want to know the scattered field. Like antenna problems, we are not usually interested in the near field, but in the far field. Because of this, we don't compute the scattered field at a specific point or points, but rather the r-independent "scattering amplitude" of the scattered field. The scattering amplitude is analogous to the gain of a radiating antenna, except that is defined from the electric field instead of the power.

The scattering amplitude is a function of the direction of the incident plane wave and the direction of the scattered field. It is defined by

$$S(\hat{k}^i, \hat{k}^s) = \lim_{r\to\infty} kr\, e^{-ikr} \frac{E^s(r\hat{k}_s)}{E^i}$$

where E^s is the amplitude of the electric field in a particular direction (usually E^s_θ or E^s_ϕ). S is in general a complex number.



Often we use another quantity, the scattering cross section, or radar cross section (RCS). This is defined to be

$$\sigma(\hat{k}^i, \hat{k}^s) = \frac{4\pi}{k^2}|S|^2 = \lim_{r\to\infty} 4\pi r^2 \frac{|E^s|^2}{|E^i|^2}$$

Since S is dimensionless, σ has units of area. This quantity has a physical interpretation:

RCS: The area that receives the amount of incident power that when radiated isotropically yields the same power density as the scattered field in a given direction.

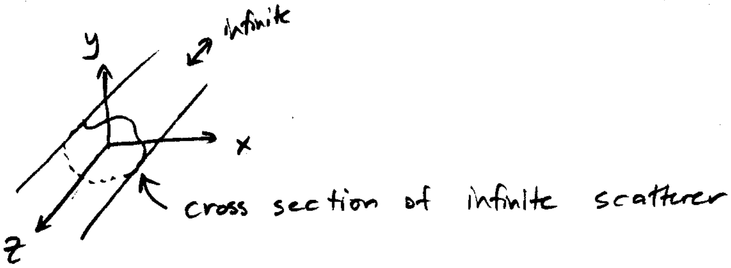
The RCS is a function of both incidence and scattered angle, so that a given object has a different "effective area" for different look angles $\hat{k}_s$ and $\hat{k}_i$.

If $\hat{k}^s = -\hat{k}^i$, then σ is the backscattering cross section, and measures the energy scattered back towards the source of the incident field. This is important in radar, as noted earlier, since the transmit and receive antennas are often in the same location.

When a stealth fighter is described as the same size as a golfball on radar, this means that the backscattering cross section is the same as that of a golf-ball-sized PEC sphere for a given range of incident directions.

2D Problems

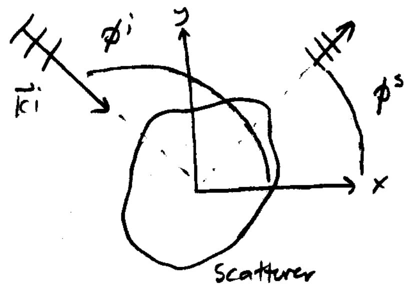
Some EM problems have an invariant direction. Examples include infinite cylinders, waveguides, straight wires, and so forth:



If the incident field wavevector has no z component, then the fields are z-invariant, and the problem becomes easier to solve than a true 3D problem. For 2D problems, there are two possible polarizations: TM, with the E field in the z direction, and TE, with the E field in the x-y plane. The two polarizations are said to be decoupled, since an incident field of one polarization produces zero scattered field of the other polarization.

For 2D problems, the scattering amplitude is defined by

$$S(\phi^i, \phi^s) = \lim_{\rho \to \infty} \sqrt{\frac{\pi k \rho}{-2i}}\, e^{-ik\rho} \frac{E^s(\phi^s)}{E^i}$$



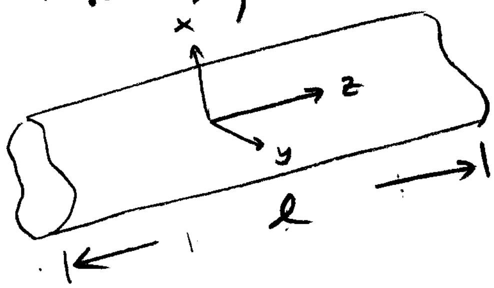
In 2D, the quantity corresponding to the RCS is the scattering width (SW), which is

$$\sigma_{2D} = \frac{4}{k}|S|^2 = \lim_{\rho \to \infty} 2\pi\rho \frac{|E^s|^2}{|E^i|^2}$$

and has units of length.

Conversion between 2D and 3D

It is possible to convert between the 2D scattering width and the 3D RCS: For a section of a long cylindrical object of length ℓ,



$$\sigma_{3D} \simeq \sigma_{2D} \frac{2\ell^2}{\lambda}$$

where the relationship is approximate due to diffraction at the ends of the object.

Scattering coefficient

It is often convenient to normalize the RCS or SW by the size of the object. This is especially convenient for large surfaces illuminated by an antenna footprint, since it allows us to characterize scattering independently of the footprint area:

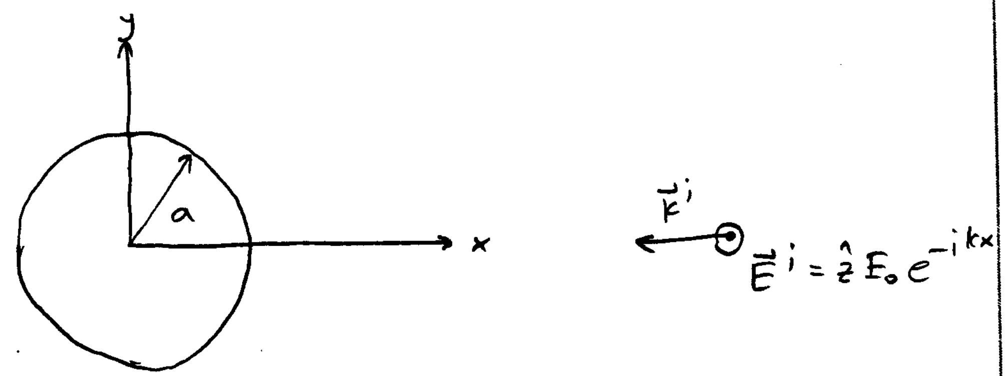
$$\sigma^0_{3D} = \frac{RCS}{Area} = \frac{\sigma_{3D}}{A}$$ (3D scattering coefficient or normalized RCS)

$$\sigma^0_{2D} = \frac{SW}{Width} = \frac{\sigma_{2D}}{W}$$ (2D scattering coefficient or normalized SW)

Consider the problem of a plane wave illuminating a PEC circular cylinder. Due to the symmetry of the scatterer, we can choose any direction for the incident field wavevector. Let's chose $\vec{k}^i = -\hat{x}$. The polarization does make a difference. Regardless of the direction of polarization, we can break it up into two components, one parallel to the axis of the cylinder, and the other perpendicular. We call the first case TM, since H is perpendicular to the axis, and the second case is TE.

TM polarization

Let's do the TM case first. If the cylinder is along the z axis, the picture is



We need to find the scattered field such that

$$\vec{E} = \vec{E}^s + \vec{E}^i$$

To do this, we will use the fact that at the PEC surface, the boundary condition is

$$\hat{n} \times \vec{E} = 0$$

So that

$$\hat{n} \times \vec{E}^s = -\hat{n} \times \vec{E}^i$$

We will use the modal approach, which means expanding the known field $\vec{E}^i$ and the unknown field $\vec{E}^s$ in the same set of orthogonal functions and matching boundary conditions at the conductor.

The natural coordinate system to use is cylindrical. We need to expand the incident plane wave in terms of cylindrical waves. This can be done using the identity

$$e^{-ikx} = e^{-ik\rho\cos\phi} = \sum_{-\infty}^{\infty} J_n(k\rho)\, e^{in\phi} (i)^{-n}$$

Using this identity,

$$\vec{E}^i(\vec{r}) = \hat{z} E_0 \sum_{n=-\infty}^{\infty} i^{-n} J_n(k\rho) e^{in\phi}$$

Since $\vec{E}^s(\vec{r})$ is a solution to Maxwell's equations, it can be expanded in terms of solutions to the Helmholtz equation. In cylindrical coordinates, these are

$$\left[A_n' J_n(k_\rho \rho) + B_n' Y_n(k_\rho \rho)\right]\left[C_n e^{in\phi} + D_n e^{-in\phi}\right]\left[E_n e^{ik_z z} + F_n e^{-ik_z z}\right]$$

or

$$\left[A_n H_n^{(1)}(k_\rho \rho) + B_n H_n^{(2)}(k_\rho \rho)\right]\left[\quad " \quad\right]\left[\quad " \quad\right]$$

We choose the second, since the Bessel functions represent standing waves, whereas the scattered field is propagating, and Hankel functions represent propagating fields. By causality, the scattered field is outgoing, and the asymptotic approximations

$$H_n^{(1)}(k_\rho \rho) \sim \sqrt{\frac{-2i}{\pi k_\rho \rho}}\; i^{-n} e^{ik_\rho \rho}$$

$$H_n^{(2)}(k_\rho \rho) \sim \sqrt{\frac{2i}{\pi k_\rho \rho}}\; i^{n} e^{-ik_\rho \rho}$$

Show that the second kind Hankel function is an incoming wave, so that $B_n = 0$. Also, the fields in this problem are independent of z, so that $k_z = 0$, and $k_\rho = \sqrt{k^2 - k_z^2} = k$. The appropriate solution form is thus

$$\vec{E}^s(\vec{r}) = \hat{z} \sum_{n=-\infty}^{\infty} A_n H_n^{(1)}(k\rho) e^{in\phi}.$$

Where we have combined the D_n and C_n coefficients into the A_n coefficients, using the identity $H_n^{(1)}(x) = (-1)^n H_{-n}(x)$ for n an integer.

Now, we apply the PEC boundary condition:

$$\hat{\rho} \times \hat{z} \sum_{-\infty}^{\infty} A_n H_n^{(1)}(ka) e^{in\phi} = -\hat{\rho} \times \hat{z} E_0 \sum_{-\infty}^{\infty} J_n(ka) e^{in\phi} (i)^{-n}$$

So that

$$\sum_{-\infty}^{\infty} A_n H_n^{(1)}(ka)\, e^{in\phi} = -\sum_{-\infty}^{\infty} J_n(ka) e^{in\phi}\, i^{-n}\, E_0$$

Since the functions $e^{in\phi}$ are orthogonal, the terms of the two series must be equal:

$$A_n H_n^{(1)}(ka) = -J_n(ka)\, i^{-n} E_0$$

Solving for A_n,

$$A_n = \frac{-J_n(ka)\, i^{-n} E_0}{H_n^{(1)}(ka)}$$

The scattered field is

$$\vec{E}_s(\vec{r}) = -\hat{z} E_0 \sum_{-\infty}^{\infty} \frac{J_n(ka)\, i^{-n}}{H_n^{(1)}(ka)} H_n^{(1)}(k\rho)\, e^{in\phi}$$

which is the desired solution to the problem.

<u>Scattering amplitude</u>

From the definition, the scattering amplitude is

$$S(\phi) = \lim_{\rho\to\infty} -\sqrt{\frac{\pi k\rho}{-2i}}\, e^{-ik\rho} \sum_{-\infty}^{\infty} \frac{J_n(ka)}{H_n^{(1)}(ka)} i^{-n} H_n^{(1)}(k\rho)\, e^{in\phi}$$

Using the asymptotic expension of the Hankel function,

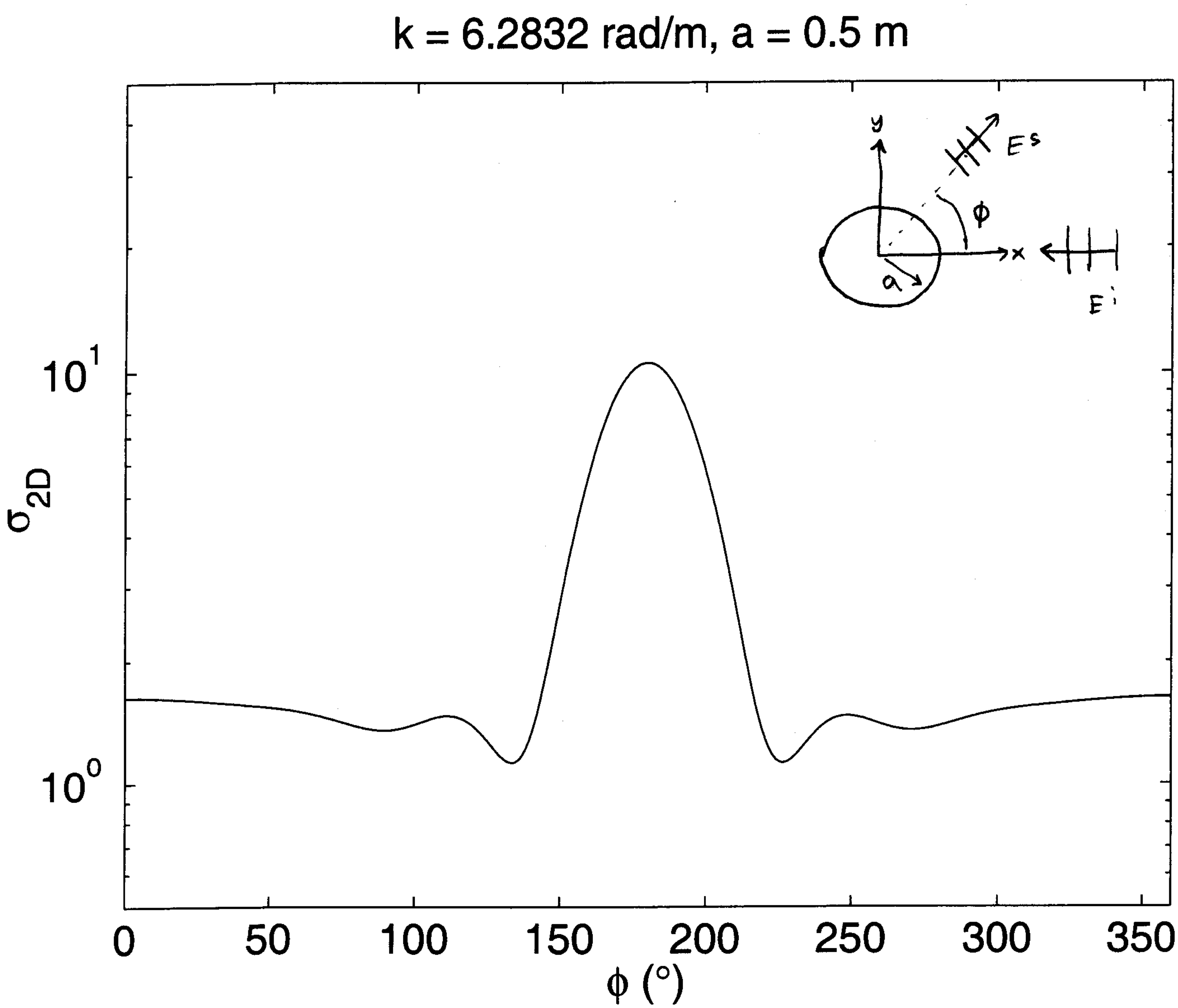
$$S(\phi) = \lim_{\rho\to\infty} -\sqrt{\frac{\pi k\rho}{-2i}}\, e^{-ik\rho} \sum_{-\infty}^{\infty} \frac{J_n(ka)}{H_n^{(1)}(ka)} i^{-n} \sqrt{\frac{-2i}{\pi k\rho}}\, e^{ik\rho - in\pi/2}\, e^{in\phi}$$

$$= -\sum_{-\infty}^{\infty} \frac{J_n(ka)}{H_n^{(1)}(ka)} e^{in\phi} (-1)^n$$

The scattering width is

$$\sigma_{2D}(\phi) = \frac{4}{k} \left| \sum_{-\infty}^{\infty} \frac{J_n(ka)}{H_n^{(1)}(ka)} (-1)^n e^{in\phi} \right|^2$$

For small and medium values of ka, $N = 100$ is more than enough for the series to converge.



k = 6.2832 rad/m, a = 0.5 m
y
E s
ϕ
x
a
E i
σ 2D
10 1
10 0
0
50
100
150
200
250
300
350
ϕ (°)