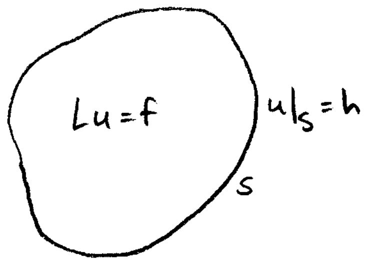
Consider a general boundary value problem:



S = boundary (possibly at ∞)
u = unknown field
f = forcing function (given)
L = partial differential operator
h = boundary value of u

Previously, we have used the 'modal method':

1. Find homogeneous solutions such that $Lu = 0$
2. Find subset of homogeneous solutions which satisfy the boundary condition, $u_n|_S = h$, $n = 0, 1, 2, \ldots$ (modes)
3. Find unique linear combination $u = \sum a_n u_n(\vec{r})$ such that $Lu = f$.

The Green's function approach is another method for solving the same BVP:

1. Find a Green's function which satisfies

$$\boxed{L\, g(\vec{r}, \vec{r}\,') = \delta(\vec{r} - \vec{r}\,')} \quad \text{and} \quad g(\vec{r}, \vec{r}\,')\big|_{\vec{r} \in S} = h$$

where $\delta(\vec{r}, \vec{r}\,')$ is a delta function located at $\vec{r}\,'$! For each $\vec{r}\,'$,

$$u(\vec{r}) = g(\vec{r}, \vec{r}\,')$$

is the field produced by a delta function or 'point source', and is the solution to the B.V.P $Lu = \delta(\vec{r} - \vec{r}\,')$, $u|_S = h$.

2. Assuming that L is linear,

$$f(\vec{r}) = f * \delta(\vec{r} - \vec{r}\,') = \int d\vec{r}\,' \, f(\vec{r}\,')\, \delta(\vec{r} - \vec{r}\,')$$

$$\equiv \int dx'\, dy'\, dz' \, f(x', y', z')\, \delta(x - x')\, \delta(y - y')\, \delta(z - z')$$

$$u(\vec{r}) = L^{-1} f = L^{-1} \int d\vec{r}\,' \, f(\vec{r}\,')\, \delta(\vec{r} - \vec{r}\,')$$

$$= \int d\vec{r}\,' \, f(\vec{r}\,')\, L^{-1} \delta(\vec{r} - \vec{r}\,')$$

$$\boxed{u(\vec{r}) = \int d\vec{r}\,' \, g(\vec{r}, \vec{r}\,')\, f(\vec{r}\,')}$$

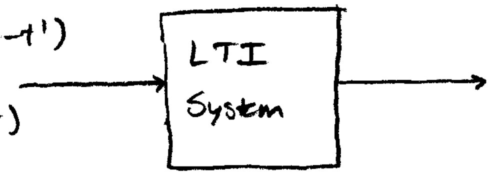
Thus, a Green's function is a very powerful tool, because we can obtain the solution to the BVP simply by integrating.

Physical interpretation:

A Green's function can be thought of as a spatial impulse response to a linear system:

$\delta(t-t')$

$x(t)$

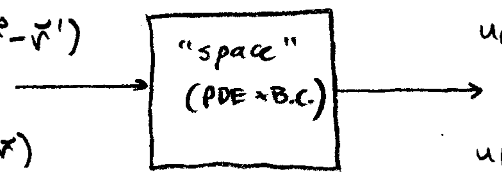


$y(t) = h(t-t')$ (impulse response)

$y(t) = x(t) * h(t)$

$\delta(\vec{r}-\vec{r}')$

$f(\vec{r})$

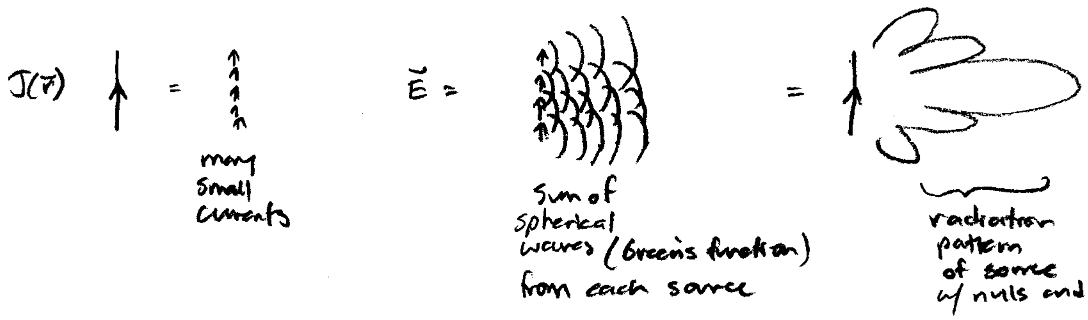


$u(\vec{r}) = g(\vec{r}, \vec{r}') = g(\vec{r}-\vec{r}')$ for translationally invariant problems

$$u(\vec{r}) = g * f = \underbrace{\int d\vec{r}' \, g(\vec{r}, \vec{r}') f(\vec{r}')}_{\text{3D convolution}}$$

= "add" up fields radiated by many infinitessimal point sources

Example: rectangular current distribution:



Consider the BVP

PDE: $\left(\frac{\partial^2}{\partial x^2} + k^2\right) u(x) = f(x)$, $\infty \le x \le \infty$

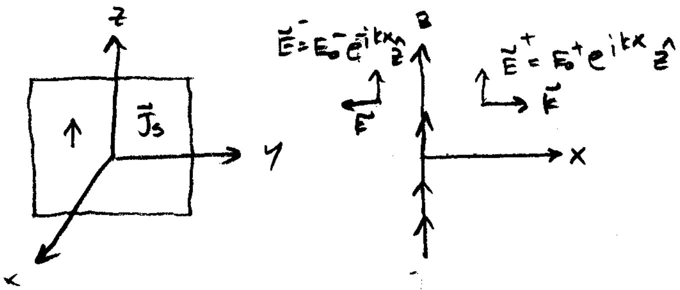
B.C: $u(x) \sim e^{ik|x|}$, $|x| \to \infty$ (outgoing waves)

What is the Green's function for this BVP?

Physically, $g(x,0)$ is the z component of the electric field radiated by a plane current at $x=0$:

$\vec{J}_s = J_0 \hat{z}$, $x=0$

$\vec{J} = J_0 \delta(x) \hat{z}$



We know that the plane current will radiate plane waves in the $\pm x$ directions. By the $\vec{E}$ boundary condition,

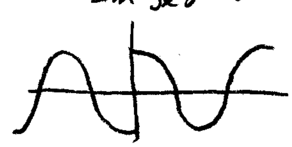
$$\hat{n} \times (\vec{E}^+ - \vec{E}^-) = 0 = \hat{x} \times (E_0^+ \hat{z} - E_0^- \hat{z})$$

$$\Rightarrow E_0^+ = E_0^- = E_0$$

Thus, $g(x,0) = \begin{cases} E_0 e^{ikx} & x \ge 0 \\ E_0 e^{-ikx} & x < 0 \end{cases}$, or $g(x,0) = E_0 e^{ik|x|}$.

Now we just need to find E_0. Using the definition of $g(x,x')$,

$$\left(\frac{\partial^2}{\partial x^2} + k^2\right) g(x,0) = \delta(x)$$



$$\frac{\partial^2}{\partial x^2} g(x,0) = \frac{\partial}{\partial x} \begin{cases} ikE_0 e^{ikx} & x > 0 \\ -ikE_0 e^{-ikx} & x < 0 \end{cases}$$

$$= \frac{\partial}{\partial x}\left(ikE_0 e^{ikx} U(x) - ikE_0 e^{-ikx} U(-x) \right)$$

$$= -k^2 E_0 e^{ikx} U(x) - k^2 E_0 e^{-ikx} U(x) + ikE_0 \delta(x) + ikE_0 \delta(x)$$

$$= -k^2 g(x,0) + 2ikE_0 \delta(x)$$

So that

$$-k^2 g(x,0) + 2ikE_0 \delta(x) + k^2 g(x,0) = \delta(x) \Rightarrow E_0 = \frac{1}{2ik}$$

Since the problem is shift invariant,

$$\boxed{g(x,x') = \frac{1}{2ik} e^{ik|x-x'|}}.$$

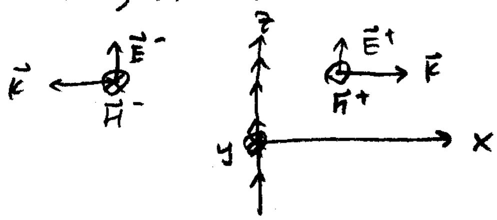
Another way to get E_0 is to use the wave equation:

$$(\nabla^2 + k^2)\vec{E} = -i\omega\mu\vec{J} = -ik\eta\vec{J}$$

For a z-directed plane current at $x=0$,

$$\left(\frac{\partial^2}{\partial x^2} + k^2\right)E_z(x) = \underbrace{-ik\eta\, J_0}_{1}\,\delta(z) \Rightarrow J_0 = \frac{-1}{ik\eta}$$

The radiated magnetic field is



$$\vec{H}^- = \hat{y}\frac{E_0}{\eta}e^{-ikx} \qquad \vec{H}^+ = -\hat{y}\frac{E_0}{\eta}e^{ikx}$$

The magnetic field B.C. is

$$\vec{J}_s = \hat{x}\times(\vec{H}^+ - \vec{H}^-) = \hat{x}\times\left(-\hat{y}\frac{E_0}{\eta} - \hat{y}\frac{E_0}{\eta}\right)$$

$$= -\hat{z}\,\frac{2E_0}{\eta} = J_0\hat{z}$$

$$-\frac{2E_0}{\eta} = -\frac{1}{ik\eta} \Rightarrow E_0 = +\frac{1}{2ik}$$

Consider the BVP

$$\text{PDE:} \quad (\nabla_{2D}^2 + k^2)\, u(x,y) = f(x,y) \quad , \quad \nabla_{2D}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\text{B.C.} \quad u(\rho) \sim \frac{e^{ik\rho}}{\sqrt{\rho}} \; , \; \rho \to \infty \quad \text{(outgoing, decaying wave)}$$

conservation of energy:

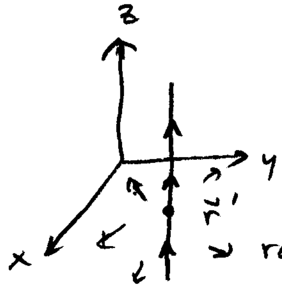
$$\int_{\text{circle}} |u(x,y)|^2 \sim 2\pi\rho |u(\rho)|^2 \to \text{constant}$$

$$\Rightarrow |u(\rho)|^2 \sim \frac{1}{\rho}$$

Let's find the Green's function for this BVP:

$$(\nabla^2 + k^2)\, g(\vec{r}, \vec{r}\,') = \delta(\vec{r} - \vec{r}\,'), \quad \vec{r} = x\hat{x} + y\hat{y}.$$

The source at $\vec{r}\,'$ represents a line source, or wire carrying a time-harmonic current:



cylindrical symmetry:
$\Rightarrow$ radiated field = cylindrical wave

Let's set $\vec{r}\,' = 0$ for convenience (we'll move it back to an arbitrary point at the end):

$$(\nabla^2 + k^2)\, g(\vec{r}, 0) = \delta(\vec{r})$$

If $\vec{r} \neq 0$, we have

$$(\nabla^2 + k^2)\, g(\vec{r}, 0) = 0 \quad , \quad g(\vec{r}, 0) = g(\rho, 0) \text{ by symmetry.}$$

We know the homogeneous solutions to this equation for cylindrical coordinates:

$$u(\rho, \phi) = \left[A_m J_m(k\rho) + B_m Y_m(k\rho) \right] \left[C_m \sin(m\phi) + D_m \cos(m\phi) \right]$$

-or-

$$= \left[A'_m \underbrace{H_m^{(1)}(k\rho)} + B'_m H_m^{(2)}(k\rho) \right] \left[C_m \sin(m\phi) + D_m \cos(m\phi) \right]$$

Hankel function:

$$\left. \begin{aligned} H_m^{(1)}(x) &= J_m(x) + iY_m(x) \sim \frac{e^{ik\rho}}{\sqrt{\rho}}, \; \rho \to \infty \end{aligned} \right\} \text{outgoing wave}$$

$$H_m^{(2)}(x) = J_m(x) - iY_m(x) \sim \frac{e^{-ik\rho}}{\sqrt{\rho}}, \; \rho \to \infty$$

By the symmetry of the source, $m = 0$. By the radiation B.C., $B_0' = 0$. Thus,

$$g(\vec{r}, 0) = A_0' H_0^{(1)}(k\rho).$$

Now we just need to find the constant A_0'. Let's integrate both sides of the defining equation over a disk centered at $(0,0)$:

$$\int_0^{2\pi}\int_0^a (\nabla^2+k^2)\, g(\vec{r},0)\, \rho\, d\rho\, d\phi = \int_0^{2\pi}\int_0^a \delta(x)\delta(y)\, \rho\, d\rho\, d\phi = 1$$

Using the divergence theorem, $\int_V \nabla\cdot\vec{A} = \oint_{bdv} \vec{A}\cdot d\vec{s}$,

$$\int_0^{2\pi}\int_0^a \nabla^2 g\, \rho\, d\rho\, d\phi = \int_0^{2\pi}\int_0^a \nabla\cdot(\nabla g)\, \rho\, d\rho\, d\phi$$

$$= \int_0^{2\pi} \nabla g \cdot \hat{\rho}\; a\, d\phi$$

If we let $a \to 0$, then

$$\int_0^{2\pi}\int_0^a k^2 g(\vec{r},0)\, \rho\, d\rho\, d\phi \to 0$$

Putting all this together,

$$\int_0^{2\pi} \left(\frac{\partial}{\partial\rho} A_0' H_0^{(1)}(k\rho)\, \hat{\rho}\right)\Big|_{\rho=a} \cdot \hat{\rho}\; a\, d\phi = 1$$

As $a \to 0$,

$$\frac{\partial}{\partial\rho} H_0^{(1)}(k\rho) = -k\, H_1(k\rho) + \overset{\nearrow 0}{\frac{0}{\rho}} H_0(k\rho)$$

using $\frac{\partial}{\partial x} Z_m(x) = -Z_{m+1}(x) + \frac{m}{x} Z_m(x)$ where Z_m is any Bessel function

$$= -k\,(\overset{\nearrow 0}{J_1(k\rho)} + i\, Y_1(k\rho))$$

$$\simeq -ik\left(\frac{2}{\pi k\rho}\right)$$

$$= -\frac{2i}{\pi\rho}$$

We now have

$$1 = \int_0^{2\pi} \left(-\frac{2i}{\pi a}\right) a\, d\phi\, A_0' = -\frac{2i}{\pi}\cdot 2\pi\, A_0' = -4i A_0'$$

$$\Rightarrow A_0' = -\frac{1}{4i} = \frac{i}{4}$$

So that $g(\vec{r},0) = \frac{i}{4} H_0^{(1)}(k\rho)$. By translational invariance,

$$g(\vec{r},\vec{r}\,') = \frac{i}{4} H_0^{(1)}\left(k\,|\vec{r}-\vec{r}\,'|\right) \qquad (2D)$$

In EM problems, the most common Green's function is the 3D free space Green's function. This corresponds to the BVP consisting of empty space with a radiation boundary condition at infinity.

Before finding the Green's function for EM fields, let's first solve a simpler scalar problem:

$$Lu = (\nabla^2 + k^2)u$$

We want to find $g(\vec{r}, \vec{r}\,')$ such that

$$(\nabla^2 + k^2)\, g(\vec{r}, \vec{r}\,') = -\delta(\vec{r} - \vec{r}\,')$$

↳ the sign is a convention...

Since space is homogeneous, let's temporarily set $\vec{r}\,' = 0$,

$$(\nabla^2 + k^2)\, g(\vec{r}) = -\delta(\vec{r})$$

Also, g must be a function of r only, since empty space is rotationally symmetric:

$$(\nabla^2 + k^2)\, g(r) = -\delta(\vec{r})$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial g}{\partial r}\right) + k^2 g = 0 \qquad \text{if} \quad r > 0$$

Making the substitution $v(r) = r\,g(r)$ gives

$$\frac{d^2}{dr^2} v(r) + k^2 v(r) = 0$$

So that

$$v(r) = A e^{-ikr} + B e^{ikr}$$

By the radiation boundary condition, the solution must be <u>outgoing</u>, so that $A = 0$, and

$$g(r) = B\frac{e^{ikr}}{r}$$

Now we just need to find B. This is done by integrating both sides of the definition of g over a small sphere containing the origin:

$$\iiint_V dv \, (\nabla^2 + k^2) \, B\frac{e^{ikr}}{r} = \iiint_V -\delta(x)\delta(y)\delta(z) \, dv$$

$$\iiint_V \nabla^2 B\frac{e^{ikr}}{r} + \iiint dv \, B\frac{e^{ikr}}{r} k^2 = -1$$

As the radius of V becomes small, the second term on the left vanishes, due to the r^2 in $dv = r^2 \sin\theta \, dr \, d\theta \, d\phi$.

The first term on the left we integrate using the divergence theorem:

$$\iiint_V dv \nabla^2 B\frac{e^{ikr}}{r} = \iiint_V dv \nabla\cdot\left(\nabla B\frac{e^{ikr}}{r}\right)$$

$$= \oiint_S \nabla B\frac{e^{ikr}}{r} \cdot d\vec{s}$$

$$= \oiint_S \left(\frac{\partial}{\partial r} B\frac{e^{ikr}}{r}\right) r^2 \sin\theta \, d\theta \, d\phi$$

$$= \oiint_S \left(B \, ik\frac{e^{ikr}}{r} - B\frac{e^{ikr}}{r^2}\right) r^2 \sin\theta \, d\theta \, d\phi$$

$$= -B \, 4\pi \quad \text{as } r \to 0$$

Thus, $B = 1/4\pi$, and the Green's function is

$$g(r) = \frac{e^{ikr}}{4\pi r}$$

Shifting the source point back to $\vec{r}\,'$ from the origin,

$$\boxed{g(\vec{r}, \vec{r}\,') = \frac{e^{ik|\vec{r} - \vec{r}\,'|}}{4\pi|\vec{r} - \vec{r}\,'|}}$$

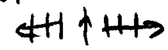
This is the scalar free space Green's function, such that

$$u(\vec{r}) = \iiint d\vec{r}\,' \, g(\vec{r}, \vec{r}\,') \, f(\vec{r}\,')$$

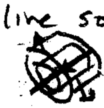
if $(\nabla^2 + k^2) u = -f$.

The scalar Green's functions for $\nabla^2 + k^2$ and the radiation boundary condition in various dimensions are

1D: $g(x,x') = \frac{1}{2ik} e^{ik|x-x'|}$ (plane source)



2D: $g(\vec{r},\vec{r}\,') = \frac{i}{4} H_0^{(1)}(k|\vec{r}-\vec{r}\,'|)$ (line source)



3D: $g(\vec{r},\vec{r}\,') = \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|}$ (point source)

We have seen how the 1D and 2D scalar Green's functions relate to vector fields. How about the 3D scalar Green's function?