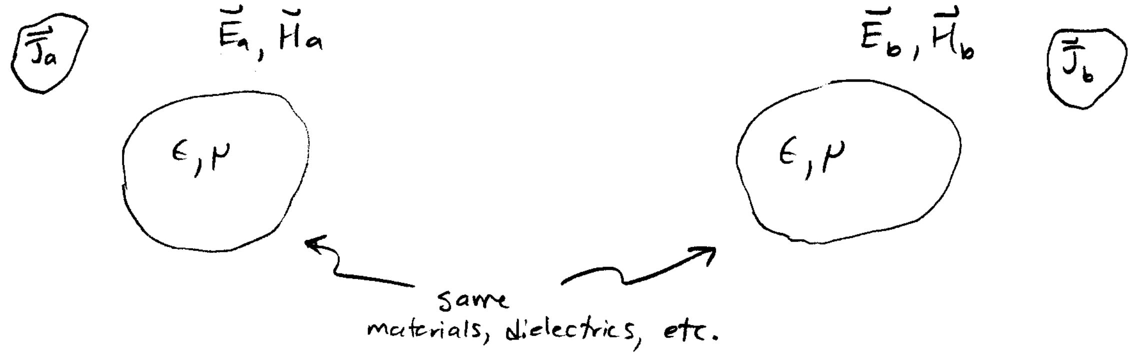
Reciprocity and reaction are concepts based on exciting a system using <u>two</u> distinct sets of sources:



The first source, $\vec{J}_a$, produces fields $\vec{E}_a$ and $\vec{H}_a$. The second source – in the same environment – produces different fields $\vec{E}_b$ and $\vec{H}_b$. The reaction between the source in picture a and the fields in picture b is

$$\langle \vec{J}_a, \vec{E}_b \rangle = \int_V dv \; \vec{J}_a \cdot \vec{E}_b$$

Note that this isn't directly a physical quantity, since it involves a source from one picture and a field from another. Think of $\langle \vec{J}_a, \vec{E}_b \rangle$ as the field that would be received by an antenna with current $\vec{J}_a$ if it could be placed in picture b without perturbing the field $\vec{E}_b$.

A system is said to be <u>reciprocal</u> if

$$\langle \vec{J}_a, \vec{E}_b \rangle = \langle \vec{J}_b, \vec{E}_a \rangle$$

for all possible sources $\vec{J}_a$ and $\vec{J}_b$.

We now prove that free space is reciprocal. In free space,

$$\langle \vec{J}_a, \vec{E}_b \rangle - \langle \vec{J}_b, \vec{E}_a \rangle = \int_V dv \left[\vec{J}_a \cdot \vec{E}_b - \vec{J}_b \cdot \vec{E}_a \right]$$

$$= \int_V dv \left[(\nabla \times \vec{H}_a + i\omega\epsilon E_a) \cdot \vec{E}_b - (\nabla \times \vec{H}_b + i\omega\epsilon E_b) \cdot \vec{E}_a \right]$$

$$= \int_V dv \left[\nabla \times \vec{H}_a \cdot \vec{E}_b - \nabla \times \vec{H}_b \cdot \vec{E}_a + \cancel{i\omega\epsilon \vec{E}_a \cdot E_b} - \cancel{i\omega\epsilon \vec{E}_b \cdot \vec{E}_a} \right]$$

$$= \int_V dv \left[\nabla \times \vec{H}_a \cdot \vec{E}_b - \nabla \times \vec{H}_b \cdot E_a + \underbrace{i\omega\mu \vec{H}_a \cdot \vec{H}_b - i\omega\mu \vec{H}_b \cdot \vec{H}_a}_{\text{zero}} \right]$$

$$= \int_V dv \left[\nabla \times \vec{H}_a \cdot E_b - \nabla \times \vec{H}_b \cdot E_a + \nabla \times E_a \cdot H_b - \nabla \times \vec{E}_b \cdot \vec{H}_a \right]$$

$$= \int_V dv \left[\underbrace{\left(\vec{H}_b \cdot \nabla\times\vec{E}_a - \vec{E}_a \cdot \nabla\times\vec{H}_b \right)}_{\nabla\cdot(\vec{E}_a\times\vec{H}_b)} - \underbrace{\left(\vec{H}_a \cdot \nabla\times\vec{E}_b - \vec{E}_b\cdot\nabla\times\vec{H}_a \right)}_{\nabla\cdot(\vec{E}_b\times\vec{H}_a)} \right]$$

$$= \int_V dv\, \nabla\cdot\left[\vec{E}_a\times\vec{H}_b - \vec{E}_b\times\vec{H}_a \right]$$

$$= \oint_S d\vec{S}\cdot\left[\vec{E}_a\times\vec{H}_b - \vec{E}_b\times\vec{H}_a \right]$$

If we let the boundary S of V go to infinity, then eventually E and H will become spherical waves with

$$\vec{H} = \hat{r}\times\vec{E}/\eta \,, \quad \hat{r}\cdot\vec{E} = \hat{r}\cdot\vec{H} = 0$$

Thus,

$$\langle \vec{J}_a, \vec{E}_b \rangle - \langle \vec{J}_b, \vec{E}_a \rangle = \oint_S ds\, \hat{r}\cdot\left[\vec{E}_a\times(\hat{r}\times\vec{E}_b/\eta) - \vec{E}_b\times(\hat{r}\times\vec{E}_a/\eta) \right]$$

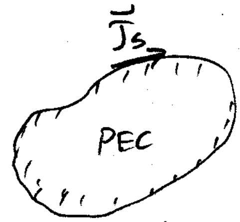
$$= \oint_S ds\, \hat{r}\cdot\left[\cancel{\frac{\vec{E}_a\cdot\vec{E}_b}{\eta}\hat{r}} - \frac{\vec{E}_a\cdot\hat{r}}{\eta}\vec{E}_b - \cancel{\frac{\vec{E}_b\cdot\vec{E}_a}{\eta}\hat{r}} + \frac{\vec{E}_b\cdot\hat{r}}{\eta}\vec{E}_a \right]$$

$$= 0 \quad \text{as } r\to\infty.$$

Thus, free space is reciprocal.

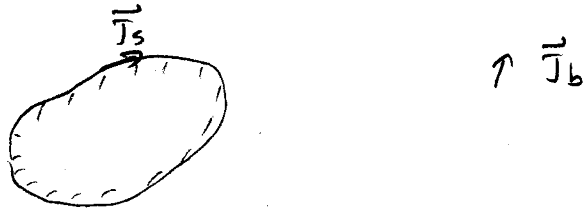
Here is an example of the use of this theorem:

Consider an impressed tangential current source at the surface of a conductor. What is the radiated field?



$\vec{E} = ?$

Denote this source as $\vec{J}_a$. Let $\vec{J}_b$ be a dipole at a point outside the PEC:



Then $\langle \vec{J}_a, \vec{E}_b \rangle = \langle \vec{J}_b, \vec{E}_a \rangle$. Since source b produces no tangential electric field at the PEC, $\langle \vec{J}_a, \vec{E}_b \rangle = 0$. Thus, by reciprocity, $\langle \vec{J}_b, \vec{E}_a \rangle = 0$ for any source $\vec{J}_b$. It follows that $\vec{E}_a = 0$. This proves that tangential current at a PEC body does not radiate! (for a flat plane, this also follows from the image theorem.)

It also follows from this theorem that an antenna in a reciprocal medium has a receiving pattern that is identical to its radiation pattern.

For an arbitrary medium with permeability and permittivity matrices $\bar{\bar{\epsilon}}$ and $\bar{\bar{\mu}}$, it can be shown that a medium is reciprocal if and only if

$$\left.\begin{aligned} \bar{\bar{\epsilon}} &= \bar{\bar{\epsilon}}^T \\ \bar{\bar{\mu}} &= \bar{\bar{\mu}}^T \end{aligned}\right\} \text{ reciprocal material}$$

or that the matrices are symmetric.

Contrast this with the condition for lossless media:

$$\left.\begin{aligned} \bar{\bar{\epsilon}} &= \bar{\bar{\epsilon}}^\dagger \\ \bar{\bar{\mu}} &= \bar{\bar{\mu}}^\dagger \end{aligned}\right\} \text{ lossless material}$$

where the dagger denotes conjugate transpose: $A^\dagger \equiv (A^*)^T$.

A gyrotropic material is lossless but nonreciprocal:

$$\bar{\bar{\mu}} = \begin{bmatrix} \mu & i\mu_g & 0 \\ -i\mu_g & \mu & 0 \\ 0 & 0 & \mu_z \end{bmatrix} \Rightarrow \begin{aligned} &\bar{\bar{\mu}} \neq \bar{\bar{\mu}}^T \rightarrow \text{nonreciprocal} \\ &\bar{\bar{\mu}} = \mu^\dagger \rightarrow \text{lossless} \end{aligned}$$

This is why we can build such devices as isolators which have different S parameters in one direction than another.

A conductor is lossy but reciprocal:

$$\bar{\bar{\epsilon}} = \left(\epsilon - \sigma/i\omega\right)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{aligned} &\bar{\bar{\epsilon}} = \bar{\bar{\epsilon}}^T \rightarrow \text{reciprocal} \\ &\bar{\bar{\epsilon}} \neq \bar{\bar{\epsilon}}^\dagger \rightarrow \text{lossy} \end{aligned}$$

$-i\omega\epsilon E + \sigma E$