QUESTION

The wave equation for $\vec{E}$ is

$$(\nabla^2 + k^2)\vec{E} = -i\omega\mu\vec{J}$$

which is the scalar equation $(\nabla^2 + k^2)u = f$ for each component.

So can we say that

$$\vec{E}(\vec{r}) = \iiint d\vec{r}'\; g(\vec{r},\vec{r}')\left(-i\omega\mu\,\vec{J}(\vec{r}')\right) \quad ?$$

The answer is no. The reason is that the wave equation has more solutions than Maxwell's equations, e.g.

$$\vec{E} = \hat{x}e^{ikx}$$

satisfies $(\nabla^2 + k^2)\hat{x}e^{ikx} = 0$ but not Gauss's law.

$$\nabla\cdot\epsilon\vec{E} = \epsilon\nabla\cdot\hat{x}e^{ikx}$$
$$= \epsilon\, ik\, e^{ikx} \neq 0$$

So we have to be a bit more clever...

To find a Green's function for Maxwell's equation, we will find a vector field which also satisfies the wave equation, but from which a valid $\vec{E}$ satisfying Gauss's law can be obtained.

We will need two theorems:

If $\nabla\cdot\vec{B} = 0$, then there exists a vector field $\vec{A}$ such that $\vec{B} = \nabla\times\vec{A}$.

If $\nabla\times\vec{C} = 0$, then there is a function such that $\nabla f = \vec{C}$.

(for proofs, see a vector calculus text)

From Gauss's law $\nabla\cdot\vec{B} = 0$, by the first theorem there exists $\vec{A}$ such that

$$\vec{B} = \nabla\times\vec{A}$$

$\vec{A}$ is the <u>magnetic</u> <u>vector</u> <u>potential</u>.

From Faraday's law,

$$\nabla\times\vec{E} = i\omega\vec{B} = i\omega\nabla\times\vec{A}$$

or

$$\nabla\times\left(\vec{E} - i\omega\vec{A}\right) = 0$$

By the second theorem, there exists ϕ such that

$$\vec{E} - i\omega\vec{A} = -\nabla\phi$$

ϕ is the electric potential. Using this result in Ampere's law gives

$$\nabla\times\left(\frac{1}{\mu}\nabla\times\vec{A}\right) = -i\omega\epsilon\left(-\nabla\phi + i\omega\vec{A}\right) + \vec{J}$$

$$\nabla\times\nabla\times\vec{A} = i\omega\epsilon\mu\nabla\phi + \omega^2\epsilon\mu\vec{A} + \mu\vec{J}$$

$$-\nabla\times\nabla\times\vec{A} + k^2\vec{A} = -i\omega\epsilon\mu\nabla\phi - \mu\vec{J}$$

Using $\nabla^2\vec{A} = -\nabla\times\nabla\times\vec{A} + \nabla\nabla\cdot\vec{A}$,

$$(\nabla^2 + k^2)\vec{A} = -i\omega\epsilon\mu\nabla\phi + \nabla\nabla\cdot\vec{A} - \mu\vec{J}$$

$$(\nabla^2 + k^2)\vec{A} = \nabla(\nabla\cdot\vec{A} - i\omega\epsilon\mu\phi) - \mu\vec{J}$$

Now, we use the fact that

$$\vec{B} = \nabla\times\vec{A}$$
$$= \nabla\times(\vec{A} + \nabla\psi)$$
$$= \nabla\times\vec{A} + \cancelto{0}{\nabla\times\nabla\psi}$$
$$= \nabla\times\vec{A} = \vec{B}$$

to add a gradient term to $\vec{A}$ such that

$$\nabla\cdot\vec{A} = i\omega\epsilon\mu\phi$$

This is known as a "gauge". There are other gauges, or specifications for the divergence of $\vec{A}$. This one is called the Lorenz gauge.

Using this gauge relationship, we have

$$\boxed{(\nabla^2 + k^2)\vec{A} = -\mu\vec{J}}$$

Now we can use the scalar Green's function for each component:

$$\vec{A} = \mu\iiint d\vec{r}'\, g(\vec{r},\vec{r}')\,\vec{J}(\vec{r}')$$

$$= \mu\iiint d\vec{r}'\,\frac{e^{ik|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}\,\vec{J}(\vec{r}')$$

The electric field is

$$\vec{E} = i\omega\vec{A} - \nabla\phi$$

$$= i\omega\vec{A} - \frac{\nabla\nabla\cdot\vec{A}}{i\omega\epsilon\mu}$$

$$= i\omega\mu\int d\vec{r}'\, g(\vec{r},\vec{r}')\vec{J}(\vec{r}') - \frac{1}{i\omega\epsilon\mu}\nabla\nabla\cdot\int g(\vec{r},\vec{r}')\vec{J}(\vec{r}')$$

$$= i\omega\mu\left[1 + \underbrace{\frac{1}{k^2}\nabla\nabla\cdot}\right]\int d\vec{r}'\, g(\vec{r},\vec{r}')\,\vec{J}(\vec{r}')$$

This term cancels out the 'non-Gaussian' solutions to the wave equation!