What is the general solution to the Helmholtz equation in the cylindrical coordinate system?

$$(\nabla^2 + k^2)\underbrace{\psi(\rho,\phi,z)}_{\text{component of vector potential } \vec{A}\text{, or } \vec{E}\ldots} = 0 \qquad (1)$$

We use the method of sepcration of variables:

$$\psi(\rho,\phi,z) = f(\rho)\,g(\phi)\,h(z)$$

So that

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\right)\psi = 0$$

becomes

$$\left[ghf'' + gh\frac{1}{\rho}f' + fh\frac{1}{\rho^2}g'' + fg\,h'' + k^2 fgh = 0\right]\frac{1}{fgh}$$

$$\frac{1}{f}f'' + \frac{1}{f\rho}f' + \frac{1}{g}\rho^2 g'' + \underbrace{\frac{h''}{h}}_{} + k^2 = 0 \qquad (2)$$

This is the only z-dependent term.
It must be constant:

$$\frac{h''}{h} = -k_z^2$$

$$\hookrightarrow h(z) = e^{\pm i k_z z}$$

Using this in (2) and multiplying by ρ^2,

$$\frac{\rho^2}{f}f'' + \frac{\rho}{f}f' + \underbrace{\frac{g''}{g}}_{} + (k^2 - k_z^2)\rho^2 = 0 \qquad (3)$$

This is the only ϕ dependent term $\therefore$ constant.

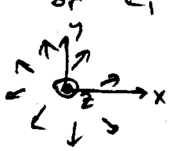
$$\frac{g''}{g} = -m^2 \Rightarrow g(\phi) = e^{\pm i m\phi}$$

Define $k_\rho^2 = k^2 - k_z^2$. Multiplying both sides of (3) by f,

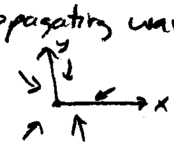
$$\rho^2 f'' + \rho f' + \left[(k_\rho\rho)^2 - m^2\right]f = 0 \quad \Big\}\ \text{Bessell's ODE}$$

$\hookrightarrow f(\rho) = A_1 J_m(k_\rho\rho) + B_1 Y_m(k_\rho\rho)$ [standing waves - like sin, cos]

or $C_1 H_m^{(1)}(k_\rho\rho) + D_1 H_m^{(2)}(k_\rho\rho)$ [propagating waves - like $e^{\pm x}$]



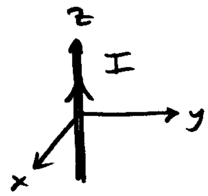
$\hookrightarrow \sim \underbrace{\dfrac{e^{ik_\rho\rho}}{\sqrt{\rho}}}_{\text{outward}\ldots}$, $\hookrightarrow \sim \underbrace{\dfrac{e^{-ik_\rho\rho}}{\sqrt{\rho}}}_{\text{inward}\ldots}$



The general solution is

$$\psi(\rho,\phi,z) = \Big[\underbrace{C_1 H_m^{(1)}(k_\rho\rho) + D_1 H_m^{(2)}(k_\rho\rho)}_{\text{or } J,\,Y}\Big]\Big[\underbrace{A_2 e^{im\phi} + B_2 e^{-im\phi}}_{\text{or sin, cos}}\Big]\Big[\underbrace{A_3 e^{ik_z z} + B_3 e^{-ik_z z}}_{\text{(or sin, cos)}}\Big]$$

Consider a time-harmonic line source on the z axis, with total current I. We wish to find the radiated fields.



The current density is $\vec{J}_s = \hat{z}\, I\, \delta(x)\,\delta(y)$, so that the radiated electric field is

$$\vec{E} = i\omega\mu \left[\overline{\overline{I}} - \frac{\nabla\nabla}{k^2} \right] \int dz' \, \frac{e^{ik|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|} \, I \, \hat{z}$$

Using the identity

$$\frac{1}{i\pi} \int dz \, \frac{e^{ikr}}{r} = H_0^{(1)}(k\rho)$$

we obtain

$$\vec{E} = i\omega\mu \left[\overline{\overline{I}} + \frac{\nabla\nabla}{k^2} \right] \frac{i}{4} H_0^{(1)}(k\rho) \, I \, \hat{z}$$

Since $\hat{z}$ is not in the longitudinal direction $(\hat{\rho})$, the $\nabla\nabla$ term is zero. Thus,

$$\vec{E} = -\frac{\omega\mu}{4} H_0^{(1)}(k\rho) \, I \, \hat{z}$$

We see that the zeroth order Hankel function is the field variation for a line source.

This problem can also be solved using the modal approach, by expanding the fields in terms of cylindrical waves and matching to the line source.

We know how to find the field due to currents in empty space. What if there are materials present? The problem of finding the fields becomes more complicated.

A homework problem in Kong shows how to find the fields for one such situation: a source over a ground plane. As the material environment becomes more complex, finding radiated fields also becomes more involved.

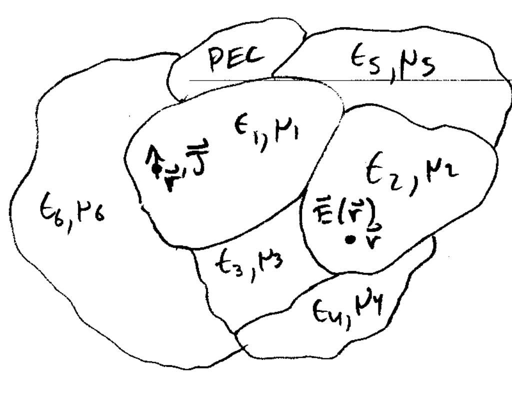
Let's look at the big picture for a minute. What are the general approaches to solving EM problems? There are several:

1. Modal method:
   - A. Find homogeneous solutions (orthogonal functions) for each region
   - B. Match boundary conditions to find modes
   - C. Match modes to source to find the unique solution
   - For complex materials, modes must be matched at all interfaces between different media ("mode matching")

2. Green's function method
   - A. Find the Green's function for a given configuration of material regions
   - B. Integrate (convolve with source) to find unique solution

3. Numerical Methods
   - Mode matching - numerical...
   - Green's functions → MOM
   - Discretize differential equations → FDTD, FEM

In order to use the Green's function method for a problem, we need to be able to find the field due to a point source of arbitrary location and orientation:

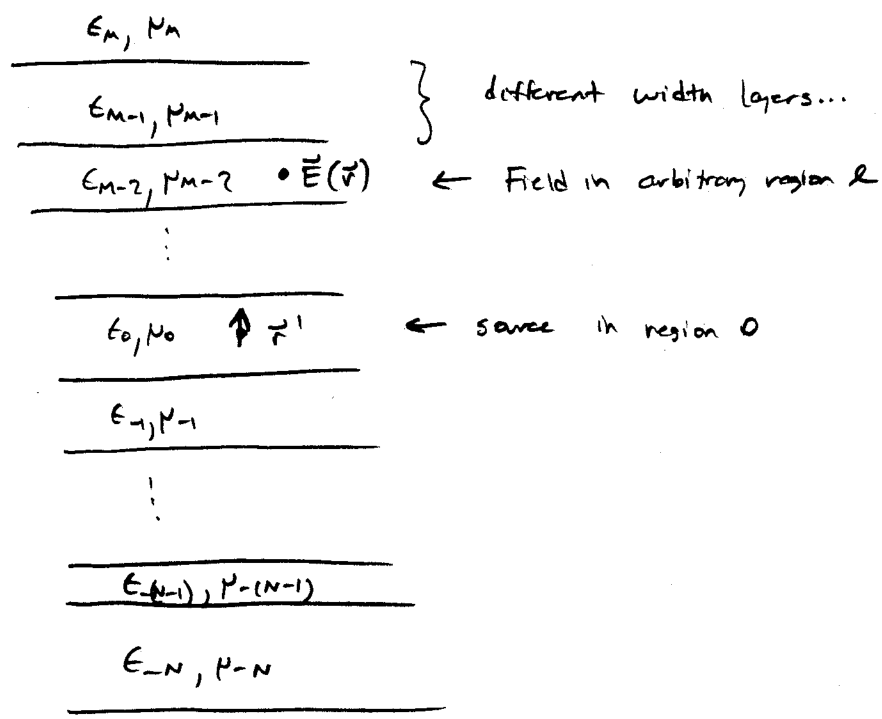


Find $\vec{E}(\vec{r})$ for point source at $\vec{r}'$

$$\downarrow$$

$$\underline{\underline{\bar{\bar{G}}(\vec{r}, \vec{r}')}}$$

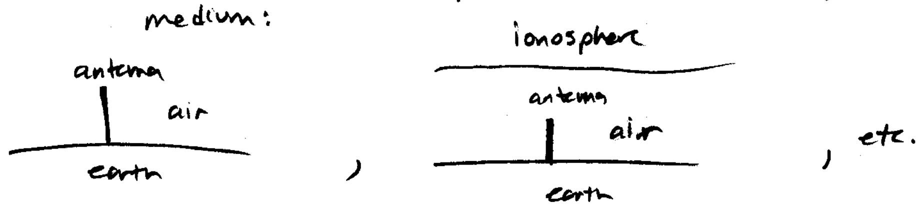
We will do this for a layered medium:



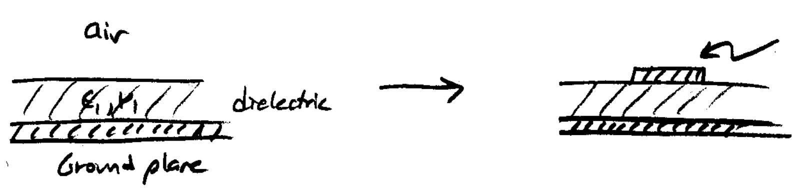
Why is this useful? For several reasons:

① Dipole antenna in layered medium

- The source is a simple model antenna. Thus, we can analyze antenna radiation in any problem which can be approximated as a layered medium:



② Analyzing microstrip and other planar circuits:

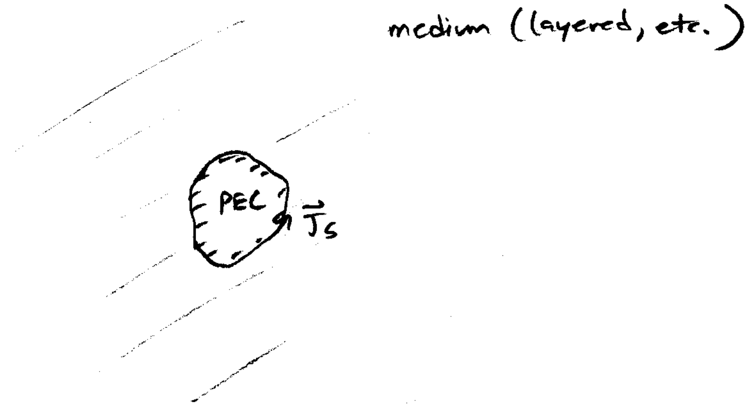


convolve current on trace with $\bar{\bar{G}}$ for this three layer medium to get $\vec{E}$ field..

③ Buried object detection

$\bar{\bar{G}}$ can be used to find radar fields reflected by the mine.

This problem is a good example of the connection between 560 material and 563 (computational EM). Any time we can find a Green's function for a given configuration of materials, we can use it as a basis for a numerical method:



$$\vec{E}_{rad} = i\omega\mu \int \bar{\bar{G}} \cdot \vec{J}_s$$

$$+$$

On the conductor surface, $\hat{n} \times (\vec{E}_{rad} + \vec{E}_{illum.}) = 0$

$$\Downarrow$$

$$\hat{n} \times \int i\omega\mu \int \bar{\bar{G}} \cdot \vec{J}_s = -\vec{E}_{illum}$$

This is an integral equation that we can solve numerically. Microstrip analysis software packages, for example, use this approach.

How do we solve this problem? As in many other situations, the first step is to use the symmetries of the problem to write down a form for the solution.

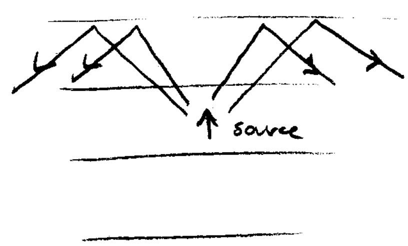
Since the material is rotationally symmetric about the z axis, we will use a cylindrical representation for the fields. We know that

$$(\nabla^2 + k^2)\vec{E}(\rho, \phi, z) = 0$$

away from the dipole. The general solution to this equation in cylindrical coordinates for one component is

$$E_z(\rho,\phi,z) = \sum_m \int dk_\rho \Big\{ A_m(k_\rho) H_m^{(1)}(k_\rho \rho)\left[E_m \sin(m\phi) + F_m \cos(m\phi)\right] e^{ik_z z}$$
$$B_m(k_\rho) H_m^{(1)}(k_\rho \rho)\left[G_m \sin(m\phi) + H_m \cos(m\phi)\right] e^{-ik_z z}$$
$$+ \text{ terms } w/\ H_m^{(2)} \Big\}$$

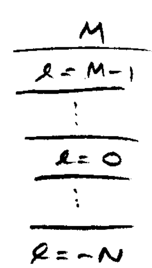
Now, we don't need the $H_m^{(2)}$ terms, since $H_m^{(2)}(k_\rho \rho) \sim \sqrt{\frac{2i}{\pi k_\rho}}\, e^{-ik_\rho \rho}$ represents a wave traveling in the $-\rho$ direction, towards the dipole. Since there are no reflections back towards the dipole, these terms will be zero:



The Hankel functions are defined by $H_m^{(1)} = J_m + iY_m$ and $H_m^{(2)} = J_m - iY_m$ in terms of the Bessel and Neumann functions.

The magnetic field will have a similar form, but with different constants than $\vec{E}$.

The next step is to use this form for the fields in each layer of the medium,



$$E_{\ell z} = \sum_m \int dk_\rho \left\{ A_{\ell m}(k_\rho)\, E_{\ell m}(\phi)\, e^{ik_z z} + B_{\ell m}(k_\rho)\, F_{\ell m}(\phi)\, e^{-ik_z z} \right\} H_m^{(1)}(k_\rho \rho)$$

$$H_{\ell z} = \sum_m \int dk_\rho \left\{ C_{\ell m}(k_\rho)\, G_{\ell m}(\phi)\, e^{ik_z z} + D_{\ell m}(k_\rho)\, H_{\ell m}(\phi)\, e^{-ik_z z} \right\} H_m^{(1)}(k_\rho \rho)$$

where the ϕ dependence is lumped into $E_{\ell m}$, $F_{\ell m}$, $G_{\ell m}$, and $H_{\ell m}$. Note that in the top and bottom regions,

$$B_{Mm} = D_{Mm} = 0$$

$$A_{-Nm} = C_{-Nm} = 0$$

Since there are only upward going waves in the top layer and downward going waves in the bottom layer.

Now let's use another symmetry. For the problem we will look at, the dipole is vertical, so the fields will be symmetric in ϕ. Thus, all the terms in the sums over m will be zero except for $m = 0$! Now,

$$E_{\ell z} = \int dk_\rho \left\{ A_{\ell 0}(k_\rho)\, e^{ik_z z} + B_{\ell 0}(k_\rho)\, e^{-ik_z z} \right\} H_0^{(1)}(k_\rho \rho)$$

$$H_{\ell z} = \int dk_\rho \left\{ C_{\ell 0}(k_\rho)\, e^{ik_z z} + D_{\ell 0}(k_\rho)\, e^{-ik_z z} \right\} H_0^{(1)}(k_\rho \rho)$$

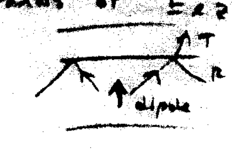
let's check the number of unknowns. There are $M+N+1$ regions, so we have

$$4(M+N+1) - 4 \quad \text{unknowns}$$

Boundary conditions at the $M+N$ interfaces give

$$4(M+N) \quad \text{equations}$$

There are enough equations to get the fields in all regions once we determine the fields in region $\ell = 0$. But most of this work has already been done in Kong, Sec. 3.4 in determining plane wave reflection coefficients for a layered medium, since the integrands of $E_{\ell z}$ and $H_{\ell z}$ behave like plane waves at a point:



In order to match to the source, we will find A_ℓ, B_ℓ, C_ℓ, and D_ℓ for no layers ($M=N=0$) and then add reflections due to the layers back in. The reflection coefficients will already have the boundary conditions taken into account.

<u>Free space</u> ($M=N=0$), Vertical electric dipole (VED)

$$E_z = \hat{z}\cdot i\omega\mu\left[\bar{\bar{I}} + \frac{\nabla\nabla}{k^2}\right]\int d\vec{r}\,'\,\frac{e^{ik|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|}\underbrace{\hat{z}I\ell\,\delta(\vec{r}\,')}_{\text{VED}}$$

$$= i\omega\mu\left[1+\frac{1}{k^2}\frac{\partial^2}{\partial z^2}\right]\frac{e^{ikr}}{4\pi r}I\ell$$

Using the "Sommerfeld identity", Eq. (4.7.15) in Kong,

$$E_z = i\omega\mu\left[1+\frac{1}{k^2}\frac{\partial^2}{\partial z^2}\right]\underbrace{\frac{i}{8\pi}\int_{-\infty}^{\infty}dk_\rho\,\frac{k_\rho}{k_z}H_0^{(1)}(k_\rho\rho)\,e^{ik_z|z|}}_{\substack{\text{Spherical wave = sum}\\ \text{of cylindrical waves...}}}I\ell$$

For $z>0$,

$$E_z = \int dk_\rho\left[\frac{-\omega\mu I\ell}{8\pi}\right]\frac{k_\rho}{k_z}H_0^{(1)}(k_\rho\rho)\underbrace{\left[1-\frac{k_z^2}{k^2}\right]}_{\frac{k^2-k_z^2}{k^2}=\frac{k_\rho^2}{k^2}}e^{ik_z|z|}$$

$$= \int dk_\rho\left[-\frac{\omega\mu I\ell\,k_\rho^3}{8\pi k^2 k_z}\right]H_0^{(1)}(k_\rho\rho)e^{ik_z|z|}$$

$$= \int dk_\rho\left[\frac{I\ell\,k_\rho^3}{8\pi\omega\epsilon_0 k_z}\right]H_0^{(1)}(k_\rho\rho)e^{ik_z|z|}$$

Now we can get $A_0(k_\rho)$ in the expression for E_{0z} by comparing the two expressions. We get

$$A_0(k_\rho) = \frac{I\ell\,k_\rho^3}{8\pi\omega\epsilon_0 k_z}$$

$$B_0 = 0$$

$$C_0 = D_0 = 0 \quad (\text{since } H_z = 0)$$

We will now put a dipole in a half-space (two layer medium). We will do the vertical magnetic dipole (VMD) since the treatment is a bit easier than the VED.

On p. 520, it is shown that a MD of strength $m\ell$ is equivalent to a current loop of area A and current I, with $m\ell = -i\omega\mu IA$.

By the duality theorem (which we will cover in sec. 5.1B), we can get H_z for a VMD from E_z for a VED:

$$\text{VED:} \quad E_z = \int dk_\rho \left[\frac{I\ell\, k_\rho^3}{8\pi\omega\epsilon_0 k_z} \right] H_0^{(1)}(k_\rho \rho)\, e^{ik_z|z|}$$

$$\Downarrow$$

$$\text{VMD} \quad H_z = \int dk_\rho \underbrace{\left[\frac{m\ell\, k_\rho^3}{8\pi\omega\mu_0 k_z} \right]}_{\longrightarrow \boxed{\frac{-iIA\,k_\rho^3}{8\pi k_z}}} H_0^{(1)}(k_\rho \rho)\, e^{ik_z|z|}$$

$$E_z \uparrow \text{VED}\ I\ell \xrightarrow{\text{duality}} H_z \uparrow \text{VMD}\ m\ell \xrightarrow{\text{equivalent}} \circlearrowleft \text{current loop}\ -i\omega\mu IA$$

Now, to add the half-space, all we need to do is include a reflection term under the integral:

$$\text{VMD on half-space} \quad H_z = \int dk_\rho \left[\frac{-iIA\,k_\rho^3}{8\pi k_z} \left(1 + R^{TE}\right) \right] H_0^{(1)}(k_\rho \rho)\, e^{ik_z z}$$

$\Uparrow$ reflection due to interface, TE since $\vec{E}$ in x-y plane...

Recall from 360 that

$$R = \frac{k_{0z} - k_{1z}}{k_{0z} + k_{1z}} = \frac{k_0\cos\theta_0 - k_1\cos\theta_1}{k_0\cos\theta_0 + k_1\cos\theta_1}$$

The final step is to approximate this integral in closed form far from the dipole. To do this, we need to use complex analysis and asymptotic integration ...