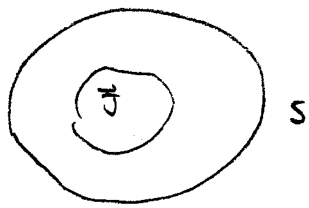
So far, we know how to find fields due to a current source in free space:

$$\vec{E}(\vec{r}) = i\omega\mu \int \bar{\bar{G}} \cdot J \, d\vec{r}\,'$$

What if we enclose the current source in a surface S?




The uniqueness theorem tells us that the field outside S is uniquely determined by $\hat{n} \times \vec{E}$ or $\hat{n} \times \vec{H}$ on S. Is there a formula for the fields outside of S in terms of $\hat{n} \times \vec{E}$ and $\hat{n} \times \vec{H}$ on S? Yes! – this is Huygens' principle.

## Derivation

First, we note that since

$$\left[-\nabla \times \nabla \times + k^2\right] \vec{E} = -i\omega\mu \vec{J}$$

the dyadic Green's function satisfies

$$\left[-\nabla \times \nabla \times + k^2\right] \bar{\bar{G}}(\vec{r}, \vec{r}\,') = -\bar{\bar{I}}\, \delta(\vec{r} - \vec{r}\,')$$

This is analogous to $(\nabla^2 + k^2) g(\vec{r}, \vec{r}\,') = -\delta(\vec{r} - \vec{r}\,')$ for the scalar Green's function. Call the differential operator $\bar{\bar{L}}$:

$$\bar{\bar{L}} = -\nabla \times \nabla \times + k^2 \bar{\bar{I}}$$

Using several vector identities, we can prove the identity

$$\int_V d\vec{r}\; \vec{E}_1 \cdot (\bar{\bar{L}} \cdot \vec{E}_2) - \int_V (\bar{\bar{L}} \cdot \vec{E}_1) \cdot \vec{E}_2 = \oint_S ds \left[ (\nabla \times \vec{E}_1) \cdot (\hat{n} \times \vec{E}_2) + \vec{E}_1 \cdot (\hat{n} \times \nabla \times \vec{E}_2) \right]$$

This is "Green's theorem" for the operator $\bar{\bar{L}} = -\nabla \times \nabla \times + k^2 \bar{\bar{I}}$.
Now, we make the substitutions $\vec{E}_1(\vec{r}) = \bar{\bar{G}}(\vec{r}, \vec{r}\,')$ and $\vec{E}_2(\vec{r}) = E(\vec{r})$:

$$\int_V d\vec{r}\; \bar{\bar{G}}(\vec{r}, \vec{r}\,') \cdot \left(-i\omega\mu \vec{J}(\vec{r})\right) - \int_V d\vec{r} \left(-\delta(\vec{r} - \vec{r}\,') \bar{\bar{I}}\right) \cdot E(\vec{r})\Big)$$

$$= \oint_S ds \left[ (\nabla \times \bar{\bar{G}}) \cdot (\hat{n} \times \vec{E}) + \bar{\bar{G}} \cdot \hat{n} \times (i\omega\mu \vec{H}) \right]$$

The second term on the left is $E(\vec{r}')$, so that

$$\vec{E}(\vec{r}') = i\omega\mu \int_V \bar{\bar{G}}(\vec{r},\vec{r}') \cdot J(\vec{r})\, d\vec{v} + \oint_S ds \left[\nabla\times\bar{\bar{G}}\cdot \hat{n}\times E_2 + i\omega\mu\bar{\bar{G}}\cdot\hat{n}\times\vec{H}\right]$$

Now, we switch $\vec{r}$ and $\vec{r}'$, and we are done:

$$\vec{E}(\vec{r}) = i\omega\mu \int_V \bar{\bar{G}}(\vec{r},\vec{r}')\cdot J(\vec{r}') + \oint_S ds' \left[\nabla\times\bar{\bar{G}}(\vec{r},\vec{r}')\cdot(\hat{n}\times\vec{E}(\vec{r}')) + i\omega\mu\,\bar{\bar{G}}(\vec{r},\vec{r}')(\hat{n}\times\vec{H}(\vec{r}'))\right]$$

where we have used $\bar{\bar{G}}(\vec{r},\vec{r}') = \bar{\bar{G}}(\vec{r}',\vec{r})$. The volume V is the outside of S, so the volume integral term is actually fields to to current sources outside of S, not the source inside S. The surface integral term is Huygens' principle, since it gives fields outside S in terms of tangential fields on the surface S.

Operator theory

In terms of operator theory, this is a fairly general derivation:

1. Write down definition of symmetric operator:

$$\langle u_1, Lu_2\rangle - \langle Lu_1, u_2\rangle$$

2. Find difference as surface integral (a "Green's theorem")

$$\langle u_1, Lu_2\rangle - \langle Lu_1, u_2\rangle = \oint_S P(u_1,u_2)\, dS$$

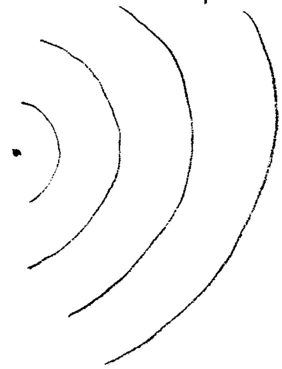
3. Substitute $u_1 = g(\vec{r},\vec{r}')$ and $u_2 = u$ and use $Lg = \delta(\vec{r}-\vec{r}')$:

$$u = \langle g, f\rangle - \int_S P(g,u)$$

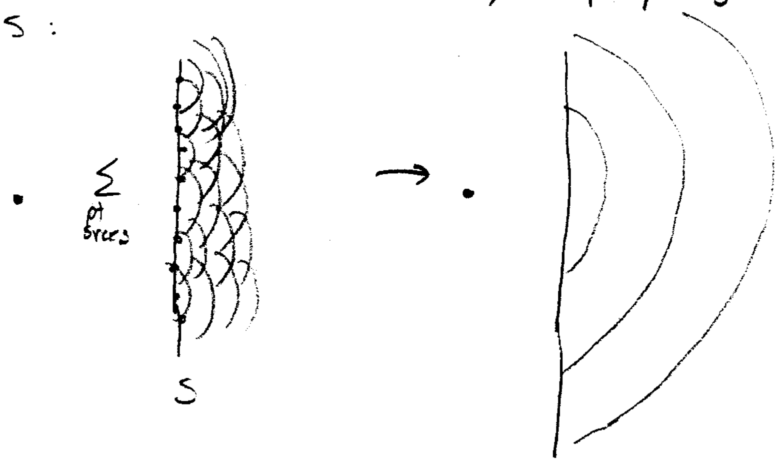
This solves the PDE $Lu = f$.

Physical meaning

Suppose we have a point source which radiates a spherical wave:



If we put a surface S in front of the source, we can obtain the same wave front by superposing many point sources along S:




Each point along S can be viewed as a new point source with strength given by the original point source to the left of S.

The surface integral we derived does this sum over point sources in the integral over $\vec{r}\,'$:

$$E(\vec{r}) = \oint_S \left[ \nabla\times\overline{\overline{G}}(\vec{r},\vec{r}\,')\, \underbrace{\hat{n}\times E(\vec{r}\,')} + i\omega\mu\overline{\overline{G}}(\vec{r},\vec{r}\,')\,\underbrace{(\hat{n}\times\vec{H}(\vec{r}\,'))} \right] d\vec{r}\,'$$

Strength of "new" point sources

"new" point sources: field $= \overline{\overline{G}} =$ field due to a point source at $\vec{r}\,'$

There are other forms of the surface integral term of the radiation integral. The basic idea is to substitute for the dyadic Green's function in terms of the scalar Green's function:

$$\overline{\overline{G}} = \left[\overline{\overline{I}} + \frac{1}{k^2}\nabla\nabla\right] g$$

When doing actual computations, it is almost always easier to use a form of the radiation integral which is in terms of $g$ rather than $\overline{\overline{G}}$.

There are two "named" formulas of this type:

Stratton-Chu:

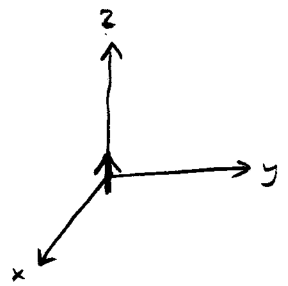
$$\vec{E}(\vec{r}) = \oint_S ds' \Big[ i\omega\mu\left[\hat{n} \times \vec{H}(\vec{r}')\right] g(\vec{r},\vec{r}') + \left[\hat{n}\cdot\vec{E}\right]\nabla' g(\vec{r},\vec{r}')$$
$$+ \left[\hat{n} \times \vec{E}(\vec{r}')\right] \times \nabla g'(\vec{r},\vec{r}') \Big]$$

Franz formula:

$$\vec{E}(\vec{r}) = \nabla\times \oint_S ds' \Big[\left[\hat{n} \times \vec{E}(\vec{r}')\right] g(\vec{r},\vec{r}') + \frac{i}{\omega\epsilon} \nabla\times\nabla\times \oint_S ds' \left[\hat{n}\times\vec{H}\right] g(\vec{r},\vec{r}')\Big]$$

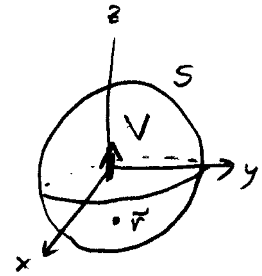
The first formula has no curl operators, which makes it a bit simpler, but the second is in terms of tangential fields only, so it is conceptually more natural.

Question. Suppose we have a point source at the origin:




$$\vec{J} = \hat{z}\, I \ell\, \delta(\vec{r})$$

Now let V be a sphere centered at the origin:




Let $\vec{r}$ be inside V. We want to find the field at $\vec{r}$.
By our previous result,

$$\vec{E}(\vec{r}) = \underbrace{i\omega\mu \int d\vec{r}\,'\, \bar{\bar{G}} \cdot \vec{J}}_{\text{Field due to the source}} + \underbrace{\oint_S ds' \left[ \nabla \times \bar{\bar{G}} \cdot \hat{n} \times \vec{E} + i\omega\mu\, \bar{\bar{G}} \cdot \hat{n} \times \vec{H} \right]}_{???}$$

The first term gives us the standard result due to a point source. But what about the second term? $\hat{n} \times \vec{E}$ and $\hat{n} \times \vec{H}$ are nonzero on S, so doesn't the surface term contribute something???

Answer: The surface term integrates to zero.

If the region of interest were outside the sphere, then the first term becomes zero, and the surface term integrates to the field due to a point source!!