With the Fresnel and Fraunhofer approximations, we began with the integral or Green's function solution to Maxwell's equations and made approximations to those expressions. Another approach to obtaining approximate solutions is to make approximations to the PDE itself.

Suppose the electric field $\vec{E}$ is in a single direction in a region of space. The amplitude of $\vec{E}$ then satisfies

$$(\nabla^2 + k^2)\, U(\vec{r}) = 0$$

U can also be thought of as a scalar field. We now assume that the field is propagating in the z direction, so that

$$U(\vec{r}) = u(\vec{r})\, e^{ikz}$$

Substituting this into the Helmholtz equation gives

$$0 = (\nabla^2 + k^2)\, u e^{ikz} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \cancel{(ik)^2 u e^{ikz}}$$

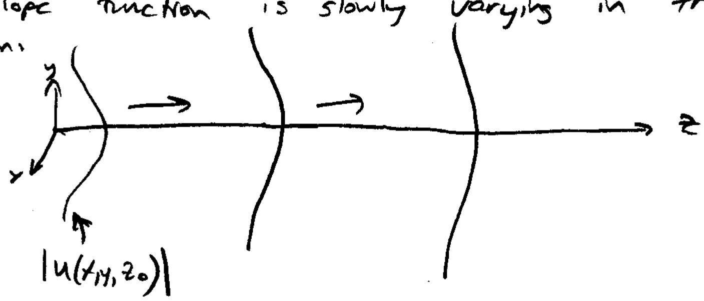
$$+ 2ik \frac{\partial u}{\partial z} e^{ikz} + \frac{\partial^2 u}{\partial z^2} e^{ikz} + \cancel{k^2 u e^{ikz}}$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2ik \frac{\partial u}{\partial z} + \underbrace{\left(\frac{\partial^2 u}{\partial z^2} e^{ikz}\right)}_{\text{assume small...}}$$

The "paraxial approximation" is that the envelope function $u(\vec{r})$ is slowly varying in the z direction, so that $\frac{\partial^2 u}{\partial z^2} \to 0$. This gives

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2ik \frac{\partial u}{\partial z} = 0$$

This equation is the "paraxial wave equation" or "parabolic wave equation." It is similar to the Schrodinger equation. The physical meaning of the paraxial approximation is that the amplitude or envelope function is slowly varying in the direction of propagation.



(a slowly spreading beam...)

The "paraboloidal wave" is a solution to the paraxial wave equation:

$$u(\vec{r}) = \frac{e^{ik\rho^2/2z}}{z}$$

This wave decays in the z direction, and has constant amplitude in the x-y plane. It is not a beam.

Mathematically, we can transform the paraboloidal wave into another solution, by setting

$$z \Rightarrow z - iz_0$$

so that

$$u(\vec{r}) = \frac{e^{ik\rho^2/2(z-iz_0)}}{(z-iz_0)} \qquad (*)$$

To understand the behavior of this new solution, we need to transform the complex terms into their real and imaginary parts:

$$\frac{1}{z-iz_0} = \underbrace{\frac{1}{R(z)}}_{\text{real}} + i\underbrace{\frac{\lambda}{\pi W^2(z)}}_{\text{imag.}}$$

$$= \frac{z+iz_0}{(z-iz_0)(z+iz_0)}$$

$$= \frac{z}{z^2+z_0^2} + i\frac{z_0}{z^2+z_0^2}$$

$$= \frac{1}{\frac{1}{z}(z^2+z_0^2)} + i\frac{1}{\frac{1}{z_0}(z^2+z_0^2)}$$

$$= \frac{1}{\underbrace{z\left(1+\frac{z_0^2}{z^2}\right)}_{R(z)}} + i\frac{\lambda/\pi}{\left[\underbrace{\sqrt{\lambda/\pi}\sqrt{z_0}\left(1+\left(\frac{z}{z_0}\right)^2\right)^{1/2}}_{W(z)}\right]^2}$$

So that

$$R(z) = z\left[1+\left(\frac{z_0}{z}\right)^2\right]$$

$$W(z) = W_0\left[1+\left(\frac{z}{z_0}\right)^2\right]^{1/2}, \quad W_0 = \sqrt{\frac{\lambda z_0}{\pi}} = \sqrt{\frac{2z_0}{k}}$$

Using these definitions in (*) gives

$$u(\vec{r}) = e^{ik\rho^2/2\left(\frac{1}{R(z)} + \frac{i2}{kW^2(z)}\right)} \frac{i}{z_0 + iz}$$

$$= e^{+ik\rho^2/2R(z)} e^{-\rho^2/W^2(z)} \frac{i}{\sqrt{z_0^2 + z^2}} e^{-i\tan^{-1}\left(\frac{z}{z_0}\right)}$$

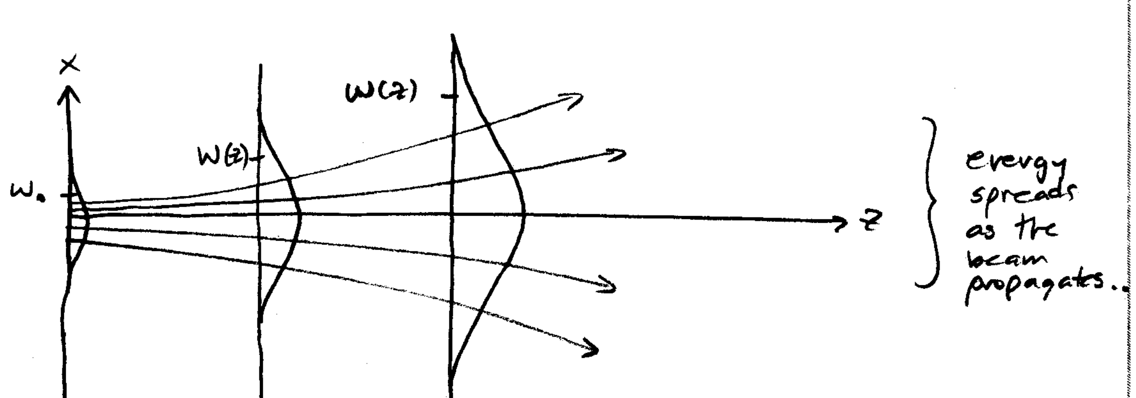
$$= \underbrace{\left(\frac{i}{z_0}\right)}_{\text{lump into amplitude constant}} \frac{W_0}{W(z)} e^{-\rho^2/W^2(z)} e^{ik\rho^2/2R(z)} e^{-i\tan^{-1}(z/z_0)}$$

$$= A \frac{W_0}{W(z)} \underbrace{e^{-\rho^2/W^2(z)}}_{\text{Gaussian amplitude in x-y plane}} \underbrace{e^{ik\rho^2/2R(z)} e^{-i\tan^{-1}\frac{z}{z_0}}}_{\text{phase terms}}$$

The function $W(z)$ is important, because it gives the width of the Gaussian beam as a function of z. Note that

$$W(z) = W_0 \sqrt{1 + \left(\frac{z}{z_0}\right)^2}$$

is increasing for $z > z_0$, so the beamwidth grows as a function of z:



z_0 is the "depth of focus", or a measure of the distance over which the beam has not spread too much. A small spot size corresponds to a short depth of focus:

$$z_0 = \frac{kW_0^2}{2}$$