A dyad is a "rank two tensor". If vectors are written in column form, then a dyad can be represented as a matrix:

$$\overline{\overline{D}}\cdot\vec{A} \Rightarrow \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

For convenience in EM, we use a different representation, that of two vectors juxtaposed without a dot or cross product between them, so that

$$\begin{aligned}\vec{A}\,\vec{B} &= (A_1\hat{x} + A_2\hat{y} + A_3\hat{z})(B_1\hat{x} + B_2\hat{y} + B_2\hat{z}) \\ &= A_1B_1\hat{x}\hat{x} + A_1B_2\hat{x}\hat{y} + \cdots\end{aligned}$$

is a dyad. The most general 3x3 dyad is a linear combination of three of these dyads:

$$\overline{\overline{D}} = \vec{A}\,\vec{B} + \vec{E}\,\vec{F} + \vec{G}\,\vec{H}$$

Now, we form combinations of dyads and vectors using the dot and cross products:

$$\vec{A}\,\vec{B}\cdot\vec{C} = \vec{A}\,(\vec{B}\cdot\vec{C}) \quad = \text{a vector}\ldots$$

$$\vec{C}\cdot(\vec{A}\,\vec{B}) = (\vec{C}\cdot\vec{A})\,\vec{B} \quad = \text{a vector}\ldots$$

$$\vec{A}\,\vec{B}\times\vec{C} = \vec{A}\,(\vec{B}\times\vec{C}) \quad = \text{a dyad}$$

and so forth. The identity dyad is

$$\overline{\overline{I}} = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z}$$

such that

$$\begin{aligned}\overline{\overline{I}}\cdot\vec{A} &= (\hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z})\cdot(A_1\hat{x} + A_2\hat{y} + A_3\hat{z}) \\ &= \hat{x}A_1 + \hat{y}A_2 + \hat{z}A_3 \\ &= \vec{A}\end{aligned}$$

as expected.

The radiation integral derived using $\vec{A}$ and ϕ gives the electric field or solution to Maxwell's equations in terms of the Green's function for a different PDE — the scalar wave equation.

By manipulating this expression, we can determine the Green's function for Maxwell's equations. This will be a 3×3 matrix of functions, or dyadic.

We start with

$$\vec{E}(\vec{r}) = i\omega\mu \left[\bar{\bar{I}} + \frac{1}{k^2} \nabla\nabla \right] \cdot \int d\vec{r}\,' \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|} J(\vec{r}\,')$$

$$= i\omega\mu \int d\vec{r}\,' \underbrace{\left\{ \left[\bar{\bar{I}} + \frac{1}{k^2} \nabla\nabla \right] \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|} \right\}}_{\text{Dyadic Green's function}} \cdot \vec{J}(\vec{r}\,')$$

$$= i\omega\mu \int d\vec{r}\,' \underbrace{\bar{\bar{G}}(\vec{r},\vec{r}\,')}_{\Uparrow} \cdot \vec{J}(\vec{r}\,')$$

3×3 matrix of functions of $\vec{r}$ and $\vec{r}\,'$

Since free space is translation-invariant, $\bar{\bar{G}}(\vec{r},\vec{r}\,')$ depends only on the difference $\vec{r}-\vec{r}\,'$, so we can rewrite this as

$$\vec{E}(\vec{r}) = i\omega\mu \int d\vec{r}\,' \bar{\bar{G}}(\vec{r}-\vec{r}\,') \cdot \vec{J}(\vec{r}\,')$$

which is a 3D convolution. Thus, we get the electric field $\vec{E}$ by convolving the dyadic Green's function with the current source $\vec{J}$.

Note: $\bar{\bar{G}}$ is very strongly singular. In fact, the integral above is a singular value integral of the third kind... the consequences of this are an advanced EM topic.

From Weiglhofer (1989),

$$\bar{\bar{G}} = \frac{e^{ikr}}{4\pi r}\Big[\big(\overbrace{k^2 + \frac{ik}{r} - \frac{1}{r^2}}^{A}\big)\underbrace{\bar{\bar{I}} + \big(}_{k^2}\overbrace{\frac{3}{r^2} - \frac{3ik}{r} - k^2}^{B}\big)\hat{r}\hat{r}\Big]\Bigg|_{\hat{r} \equiv \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|}}$$

For a z-directed diple,

$$\vec{E}(\vec{r}) = i\omega\mu \int \bar{\bar{G}}\cdot\big(I\ell\,\hat{z}\,\delta(\vec{r}')\big)\,d\vec{r}'$$

$$= i\omega\mu\,\frac{e^{ikr}}{4\pi r}\,I\ell\,\frac{1}{k^2}\Big[A\,\hat{z} + B\,\hat{r}\,(\hat{z}\cdot\hat{r})\Big]$$

$$\hat{z}\cdot\hat{r} = \cos\theta\ , \quad \hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$$

$$= i\omega\mu\,\frac{e^{ikr}}{4\pi r}\,I\ell\,\frac{1}{k^2}\Big[A\,(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}) + B\,\hat{r}\cos\theta\Big]$$

$$= \;''\quad ''\quad \Big[(A+B)\cos\theta\,\hat{r} - \sin\theta\,A\,\hat{\theta}\Big]$$

$$= \;''\quad ''\quad \Big[\Big(-\frac{2ik}{r} + \frac{2}{r^2}\Big)\cos\theta\,\hat{r} - \Big(k^2 + \frac{ik}{r} - \frac{1}{r^2}\Big)\sin\theta\,\hat{\theta}\Big]$$

$$= -i\omega\mu\,\frac{e^{ikr}}{4\pi r}\,I\ell\Big[\cancel{\hat{r}}\Big(+\frac{2i}{kr} - \frac{1}{k^2r^2}\Big)2\cos\theta\,\hat{r} + \Big(1 + \frac{i}{kr} - \frac{1}{k^2r^2}\Big)\sin\theta\,\hat{\theta}\Big]$$

$$= -i\omega\mu\,\frac{e^{ikr}}{4\pi r}\,I\ell\Big[\cancel{\hat{r}}\Big(\frac{i}{kr} + \underbrace{\Big(\frac{i}{kr}\Big)^2}\Big)2\cos\theta\,\hat{r} + \hat{\theta}\Big(1 + \frac{i}{kr} + \underbrace{\Big(\frac{i}{kr}\Big)^2}\Big)\sin\theta\Big]$$

← near fields... ↗

which agrees with (4.3.11). (Kong)

As $r \to \infty$,

$$\vec{E}(\vec{r}) \simeq -i\omega\mu\,\frac{e^{ikr}}{4\pi r}\,I\ell\,\hat{\theta}\sin\theta\,.$$

$$\vec{H}(\vec{r}) \simeq -\frac{i\omega\mu}{\eta}\,\frac{e^{ikr}}{4\pi r}\,I\ell\,\hat{\phi}\sin\theta$$

$$\vec{S}(\vec{r}) \simeq \hat{r}\,\frac{1}{\eta}\Big(\frac{\omega\mu I\ell}{4\pi r}\Big)^2\sin^2\theta = \hat{r}\,\eta\Big(\frac{kI\ell}{4\pi r}\Big)^2\sin^2\theta = \hat{r}\,\underbrace{\frac{C_1}{r^2}}_{1/r^2}\,\underbrace{\sin^2\theta}_{f(\theta)}$$

$f(\theta)$ ↓ radiation pattern