

A dyad is a "rank two tensor". If vectors are written in column form, then a dyad can be represented as a matrix:

$$\vec{D} \cdot \vec{A} \Rightarrow \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

For convenience in EM, we use a different representation, that of two vectors juxtaposed without a dot or cross product between them, so that

$$\begin{aligned} \vec{A} \vec{B} &= (A_1 \hat{x} + A_2 \hat{y} + A_3 \hat{z}) (B_1 \hat{x} + B_2 \hat{y} + B_3 \hat{z}) \\ &= A_1 B_1 \hat{x} \hat{x} + A_1 B_2 \hat{x} \hat{y} + \dots \end{aligned}$$

is a dyad. The most general 3×3 dyad is a linear combination of three of these dyads:

$$\vec{D} = \vec{A} \vec{B} + \vec{E} \vec{F} + \vec{G} \vec{H}$$

Now, we form combinations of dyads and vectors using the dot and cross products:

$$\vec{A} \vec{B} \cdot \vec{C} = \vec{A} (\vec{B} \cdot \vec{C}) = \text{a vector} \dots$$

$$\vec{C} \cdot (\vec{A} \vec{B}) = (\vec{C} \cdot \vec{A}) \vec{B} = \text{a vector} \dots$$

$$\vec{A} \vec{B} \times \vec{C} = \vec{A} (\vec{B} \times \vec{C}) = \text{a dyad}$$

and so forth. The identity dyad is

$$\vec{I} = \hat{x} \hat{x} + \hat{y} \hat{y} + \hat{z} \hat{z}$$

such that

$$\begin{aligned} \vec{I} \cdot \vec{A} &= (\hat{x} \hat{x} + \hat{y} \hat{y} + \hat{z} \hat{z}) \cdot (A_1 \hat{x} + A_2 \hat{y} + A_3 \hat{z}) \\ &= \hat{x} A_1 + \hat{y} A_2 + \hat{z} A_3 \\ &= \vec{A} \end{aligned}$$

as expected.

The radiation integral derived using \vec{A} and ϕ gives the electric field or solution to Maxwell's equations in terms of the Green's function for a different PDE — the scalar wave equation.

By manipulating this expression, we can determine the Green's function for Maxwell's equations. This will be a 3×3 matrix of functions, or dyadic.

We start with

$$\vec{E}(\vec{r}) = i\omega\mu \left[\vec{I} + \frac{1}{k^2} \nabla\nabla \right] \cdot \int d\vec{r}' \frac{e^{ik|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|} \vec{J}(\vec{r}')$$

$$= i\omega\mu \int d\vec{r}' \left\{ \left[\vec{I} + \frac{1}{k^2} \nabla\nabla \right] \frac{e^{ik|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|} \right\} \cdot \vec{J}(\vec{r}')$$

Dyadic Green's function

$$= i\omega\mu \int d\vec{r}' \vec{G}(\vec{r}, \vec{r}') \cdot \vec{J}(\vec{r}')$$

↑

3×3 matrix of functions
of \vec{r} and \vec{r}'

Since free space is translation-invariant, $\vec{G}(\vec{r}, \vec{r}')$ depends only on the difference $\vec{r} - \vec{r}'$, so we can rewrite this as

$$\vec{E}(\vec{r}) = i\omega\mu \int d\vec{r}' \vec{G}(\vec{r} - \vec{r}') \cdot \vec{J}(\vec{r}')$$

which is a 3D convolution. Thus, we get the electric field \vec{E} by convolving the dyadic Green's function with the current source \vec{J} .

Note: \vec{G} is very strongly singular. In fact, the integral above is a singular value integral of the third kind... the consequences of this are an advanced EM topic.

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From Weisshofer (1989),

$$\vec{G} = \frac{e^{ikr}}{4\pi r} \left[\underbrace{\left(k^2 + \frac{ik}{r} - \frac{1}{r^2} \right)}_A \vec{E} + \underbrace{\left(\frac{3}{r^2} - \frac{3ik}{r} - k^2 \right)}_B \hat{r} \hat{r} \right] \quad \left(\hat{r} = \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|} \right)$$

For a z-directed dipole,

$$\vec{E}(\vec{r}) = \text{imp} \int \vec{G} \cdot (\text{IR} \hat{z} \delta(\vec{r}')) d\vec{r}'$$

$$= \text{imp} \frac{e^{ikr}}{4\pi r} \text{IR} \frac{1}{k^2} \left[A \hat{z} + B \hat{r} (\hat{z} \cdot \hat{r}) \right]$$

$$\hat{z} \cdot \hat{r} = \cos\theta, \quad \hat{z} = \cos\theta \hat{r} - \sin\theta \hat{\theta}$$

$$= \text{imp} \frac{e^{ikr}}{4\pi r} \text{IR} \frac{1}{k^2} \left[A (\cos\theta \hat{r} - \sin\theta \hat{\theta}) + B \hat{r} \cos\theta \right]$$

$$= \text{imp} \frac{e^{ikr}}{4\pi r} \text{IR} \frac{1}{k^2} \left[(A+B) \cos\theta \hat{r} - \sin\theta A \hat{\theta} \right]$$

$$= \text{imp} \frac{e^{ikr}}{4\pi r} \text{IR} \frac{1}{k^2} \left[\left(-\frac{2ik}{r} + \frac{2}{r^2} \right) \cos\theta \hat{r} - \left(k^2 + \frac{ik}{r} - \frac{1}{r^2} \right) \sin\theta \hat{\theta} \right]$$

$$= -\text{imp} \frac{e^{ikr}}{4\pi r} \text{IR} \left[\hat{r} \left(+\frac{2i}{kr} - \frac{1}{k^2 r^2} \right) 2\cos\theta \hat{r} + \left(1 + \frac{i}{kr} - \frac{1}{k^2 r^2} \right) \sin\theta \hat{\theta} \right]$$

$$= -\text{imp} \frac{e^{ikr}}{4\pi r} \text{IR} \left[\hat{r} \left(\frac{i}{kr} + \left(\frac{i}{kr} \right)^2 \right) 2\cos\theta \hat{r} + \hat{\theta} \left(1 + \frac{i}{kr} + \left(\frac{i}{kr} \right)^2 \right) \sin\theta \right]$$

← near fields... →

which agrees with (4.3.11). (Kong)

As $r \rightarrow \infty$,

$$\vec{E}(\vec{r}) \approx -\text{imp} \frac{e^{ikr}}{4\pi r} \text{IR} \hat{\theta} \sin\theta$$

$$\vec{H}(\vec{r}) \approx -\frac{\text{imp}}{\eta} \frac{e^{ikr}}{4\pi r} \text{IR} \hat{\phi} \sin\theta$$

$$\vec{S}(\vec{r}) \approx \hat{r} \left(\frac{\text{impIR}}{\lambda 4\pi r} \right)^2 \sin^2\theta = \hat{r} \eta \left(\frac{k\text{IR}}{4\pi r} \right)^2 \sin^2\theta = \hat{r} \frac{C_1}{r^2} \underbrace{\sin^2\theta}_{f(\theta)}$$

↓
radiation pattern