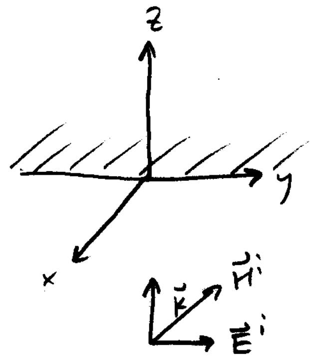
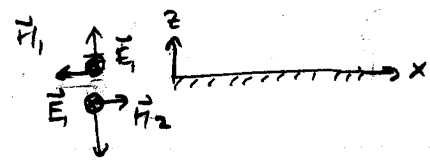
Consider a plane wave striking an infinite PEC half plane:



We use Huygens' principle, with the equivalent currents

$$\vec{J}_s = \hat{n} \times (\vec{H}_1 - \vec{H}_2)$$
$$\vec{M}_s = -\hat{n} \times (\vec{E}_1 - \vec{E}_2)$$



(Equivalent Problem 2 on p. 657)

Where

$$\vec{J}_s = 2\hat{n} \times \vec{H}$$
$$= 2\hat{z} \times \left(-\frac{1}{\eta} E \hat{x}\right)$$
$$= -\frac{2}{\eta}\hat{y} E$$

The electric field is

$$\vec{E} = \oint_S ds' \left[(\nabla \times \bar{\bar{G}}) \cdot \overset{\nearrow 0}{M_s} + i\omega\mu \bar{\bar{G}} \cdot J_s \right]$$
$$= \oint_S ds' \; i\omega\mu \; \bar{\bar{G}}(\vec{r}, \vec{r}') \cdot \left(-\frac{2}{\eta} \hat{y} E\right)$$
$$= \iint dx' dy' \left\{ i\omega\mu \left[\bar{\bar{I}} + \frac{1}{k^2} \nabla\nabla \right] g(\vec{r}, \vec{r}') \left(-\frac{2}{\eta} \hat{y} E\right) \right\}$$

There is no y' dependence in the integrand except for $g(\vec{r}, \vec{r}')$, so we use the identity

$$\int_{-\infty}^{\infty} dy' \frac{e^{ik|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|} = \frac{i}{4} H_0^{(1)}\left(k\sqrt{(x-x')^2 + z^2}\right) \qquad (y=0)$$

Also, since the fields do not depend on y, the DD term drops out, because

$$\nabla \cdot (\hat{y} E) = \frac{\partial}{\partial y} \cdots \to 0$$

We have finally

$$\vec{E}(x,z) = \hat{y} \left(\frac{2\omega\mu}{4\eta}\right) \int_{-\infty}^{\infty} dx' \, E(x') \, H_0^{(1)}\left(k\sqrt{(x-x')^2+z^2}\right)$$

Expanding the argument of the Hankel function

$$k\sqrt{(x-x')^2+z^2} = kz\left[1 + \frac{1}{2}\frac{(x-x')^2}{z^2} + \cdots\right]$$

Question. How does this differ from the far field approximation?

and using

$$H_0^{(1)}(x) \sim \sqrt{\frac{2}{i\pi x}}\, e^{ix}, \quad x \to \infty$$

we have

$$\vec{E}(x,z) \simeq \hat{y}\left(\frac{k}{2}\right)\int_{-\infty}^{\infty} E(x') \sqrt{\frac{2}{i\pi kz}}\, e^{i\left[kz + \frac{kz}{2}\frac{(x-x')^2}{z^2}\right]} dx'$$

$$= \hat{y}\,\frac{k}{2}\sqrt{\frac{2}{i\pi kz}}\, e^{ikz} \int_{-\infty}^{\infty} E(x')\, e^{ik(x-x')^2/2z}$$

For diffraction by a half-space, $E(x') = E_0$, $x' > 0$, and $E(x') = 0$ for $x' < 0$. This leads to

$$\vec{E}(x,z) = \hat{y}\,\frac{k}{2}\sqrt{\frac{2}{i\pi kz}}\, e^{ikz} \int_0^{\infty} E_0\, e^{ik(x-x')^2/2z}\, dx'$$

Now, we make the substitution

$$\pi t^2 = k(x-x')^2/z$$

so that $\sqrt{\pi}\, t = \sqrt{k/z}\,(x-x')$, $\sqrt{\pi}\, dt = -\sqrt{k/z}\, dx'$, $dx' = -\sqrt{\frac{\pi z}{k}}\, dt$, and

$$\vec{E}(x,z) = \hat{y}\,\frac{k}{2}\sqrt{\frac{2}{i\pi kz}}\, E_0 e^{ikz}\left(-\sqrt{\frac{\pi z}{k}}\right)\int_{\sqrt{\frac{k}{\pi z}}x}^{\infty} dt\, e^{i\pi t^2/2}$$

$$= \hat{y}\,\frac{E_0}{\sqrt{2i}}\, e^{ikz}\left[\underbrace{\int_{-\sqrt{\frac{k}{\pi z}}x}^{\infty} dt\, e^{i\pi t^2/2}}\right]$$

$$= \hat{y}\, \frac{E_0}{\sqrt{2i}}\, e^{ikz}\, D(x)$$

where

$$D(x) = \int_0^{\infty} dt\, e^{i\pi t^2/2} - \int_0^{-i\sqrt{\frac{k}{\pi z}}\,x} dt\, e^{i\pi t^2/2}$$

We will obtain a value for $D(x)$ using the Fresnel cosine and sine integrals:

$$C(\omega) = \int_0^{\omega} dt\, \cos\left(\pi t^2/2\right)$$

$$S(\omega) = \int_0^{\omega} dt\, \sin\left(\pi t^2/2\right)$$

so that if we define

$$F(\omega) = \int_0^{\omega} dt\, e^{i\pi t^2/2}$$

then

$$F(\omega) = C(\omega) + iS(\omega)$$

A limiting value is

$$F(\pm\infty) = \pm\tfrac{1}{2} \pm \tfrac{i}{2}$$

Now,

$$D(x) = F(\infty) - F\left(-\sqrt{\frac{k}{\pi z}}\, x\right)$$

We see that $D(-\infty) = 0$, $D(0) = \frac{1}{2} + \frac{i}{2}$, and $D(\infty) = 1 + i$. There is a graphical method for computing this function for other values of x. We plot the complex number $F(x)$ in the complex plane. From the definitions,

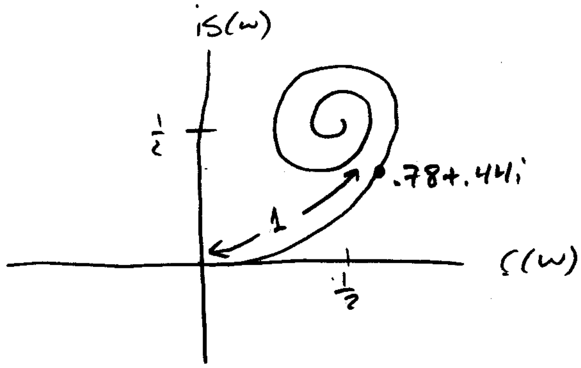
$$\frac{dC}{d\omega} = \cos\left(\frac{\pi}{2}\omega^2\right), \quad \frac{dS}{d\omega} = \sin\left(\frac{\pi}{2}\omega^2\right)$$

So that the differential path length is

$$\sqrt{(dC)^2 + (dS)^2} = \sqrt{\left(\frac{dC}{d\omega}\right)^2 + \left(\frac{dS}{d\omega}\right)^2}\, d\omega$$

$$= d\omega$$

So that ω is the length of the curve from the origin. Thus, $F(\omega)$ is the complex number on the Cornu Spiral that is a distance w from the origin, measured along the path:

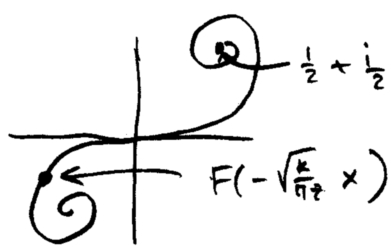


$$F(1) = .78 + .44\,i$$

The function we want is

$$D(x) = F(\infty) - F\left(-\sqrt{\tfrac{k}{\pi z}}\,x\right) = \frac{1}{2} + \frac{i}{2} - F\left(-\sqrt{\tfrac{k}{\pi z}}\,x\right)$$

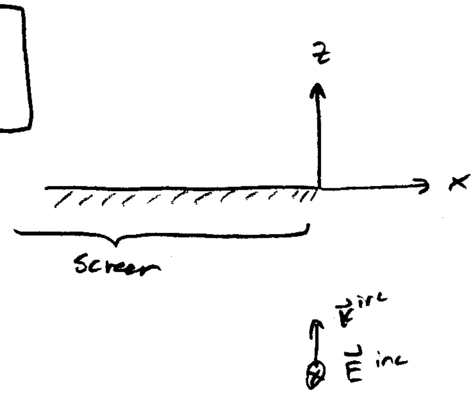
so we find $D(x)$ by subtracting two values:



$$D(x) = \frac{1}{2} + \frac{i}{2} - F\left(-\sqrt{\tfrac{k}{\pi z}}\,x\right)$$

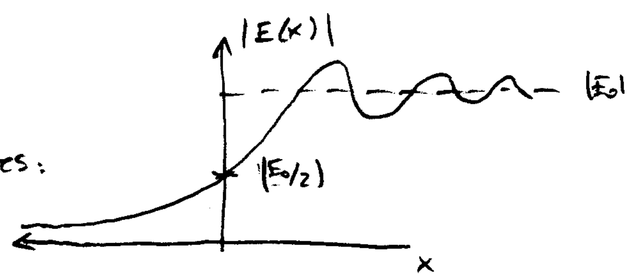
Let's look at limiting values: $D(-\infty) = 0$

$x = -\infty$
$\vec{E} = 0$
No Field



$x = \infty$
$\vec{E} = \hat{y}\,\frac{E_0}{\sqrt{2}\,i}\,e^{ikz}(1+i)$

In between, the field oscillates:



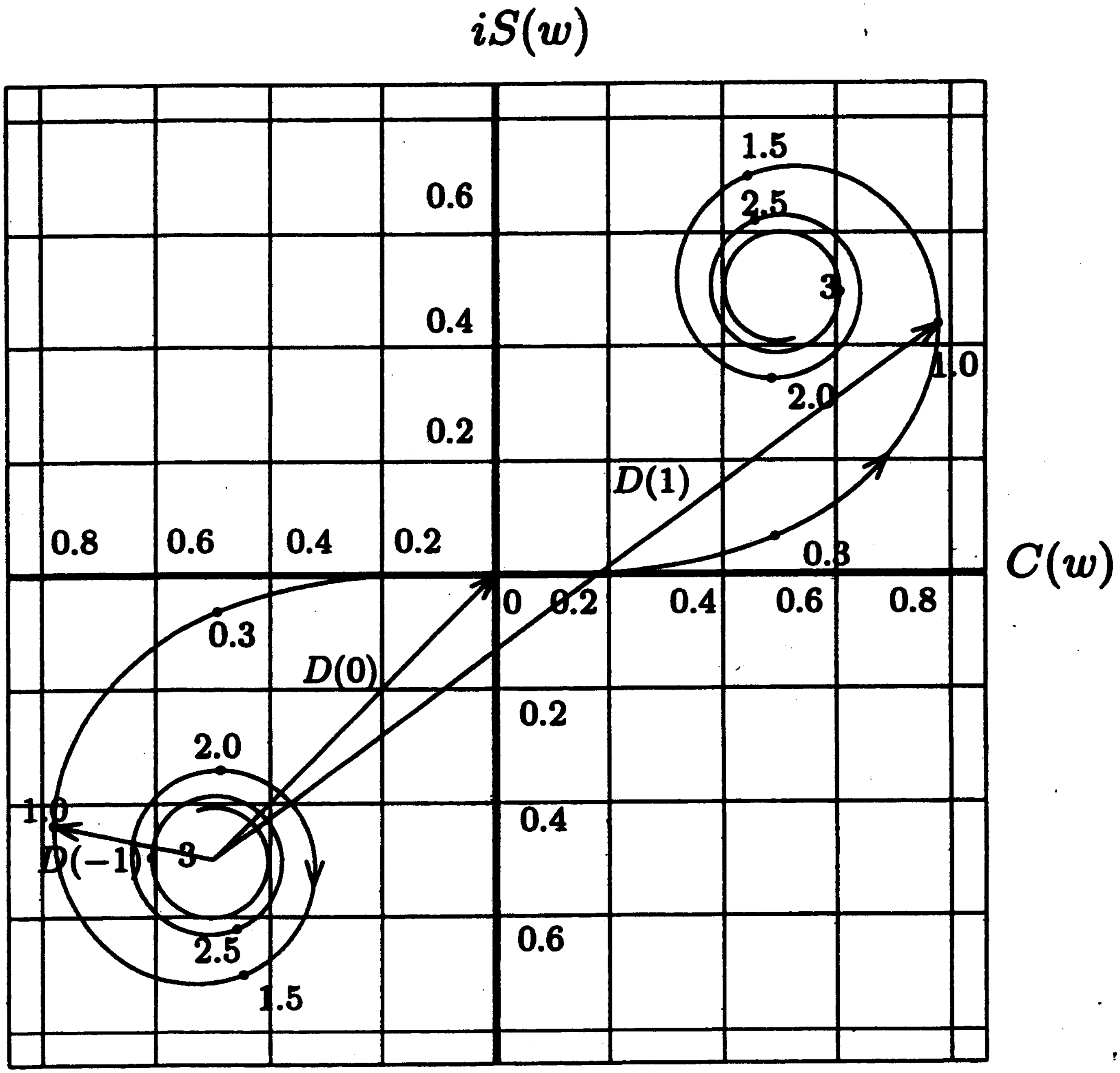


Figure 5.1D.3 Cornu spiral.

"He brought the stars nearer" - epitaph of Fraunhofer...

By making a weaker approximation to the $|\vec{r}-\vec{r}'|$ term in the radiation integral, we can further simplify the result for the diffracted field far from a finite aperture:

$$k\sqrt{(x-x')^2+z^2} \simeq kz\left[1+\frac{1}{2}\left(\frac{x-x'}{z}\right)^2+\cdots\right]$$

$$= kz\left[1+\frac{1}{2}\frac{x^2}{z^2}-\frac{xx'}{z^2}+\frac{x'^2}{2z^2}+\cdots\right]$$

$$= \underbrace{\underbrace{kz+\frac{kx^2}{2z}-\frac{kxx'}{z}}_{\text{Fraunhofer approximation}}+\frac{kx'^2}{2z}}_{\text{Fresnel approximation}}+\cdots$$

How does the Fraunhofer approximation compare to the far field approximation? We used

$$k|\vec{r}-\vec{r}'| \simeq k(r-\hat{r}\cdot\vec{r}')$$

$$= k\left(\sqrt{x^2+z^2}-\frac{x}{r}x'\right) \qquad (y=y'=z'=0)$$

$$\simeq kz\left(1+\frac{x^2}{2z^2}\right)-\frac{kx}{z}x' \qquad (z \gg x)$$

$$= kz+\frac{kx^2}{2z}-\frac{kxx'}{z}$$

which is the same as above. Thus, the far field is essentially the same as the Fraunhofer approximation, except that we are considering the near-axis region ($x \ll z$) and a y-invariant problem ($y=y'=0$) with a source on the $z'=0$ plane.

Where is the Fraunhofer approximation accurate? The first neglected term is

$$\frac{kx'^2}{2z}$$

If we require this to be small relative to 2π,

$$\frac{kx'^2}{2z} \ll 2\pi$$

$$\frac{2\pi}{\lambda}\frac{x'^2}{2z} \ll 2\pi$$

$$\boxed{\frac{x'^2}{\lambda z} \ll 1} \quad \text{(Far-field condition)}$$

What does this mean? For a given source width d, the z coordinate where we find the field must satisfy

$$z \gg \frac{d^2}{2\lambda}$$

At 3 GHz, $\lambda = 10$ cm, so that for a one wavelength object,

$$z \gg \frac{10\text{ cm}}{2}\left(\frac{10\text{cm}}{10\text{cm}}\right) = \underline{\underline{5\text{ cm}}}$$

For the 3 m dishes on the Clyde building roof, $f = 1.5$ GHz, so that $\lambda = 20$ cm, and the condition becomes

$$z \gg \frac{(3\text{ m})^2}{2(.2\text{m})} = \underline{\underline{22.5\text{ m}}} \approx 75\text{ ft}$$

How much further do we need to go beyond $d^2/2\lambda$ to be "in" the far field? The book suggests

$$z_F = 2\frac{d^2}{\lambda}$$

which becomes

$$z_F = 4\left(\frac{d^2}{2\lambda}\right) = 20\text{ cm} \qquad (1\,\lambda\text{ object})$$

$$z_F = 4\left(\frac{d^2}{2\lambda}\right) = 300\text{ ft} \qquad (3\text{ m dish @ } 1.5\text{ GHz})$$

for the two examples.

By using the Fraunhofer approximation in Eq. (5.1D.3) on p. 683 and following a derivation similar to that done for the Fresnel approximation, we obtain

$$\vec{E}(x,z) = \hat{y}\,\frac{k}{2}\sqrt{\frac{2}{i\pi kz}}\;e^{ikz + ikx^2/2z}\int_{-\infty}^{\infty} dx'\,E(x')\,e^{-i(kx/z)x'}$$

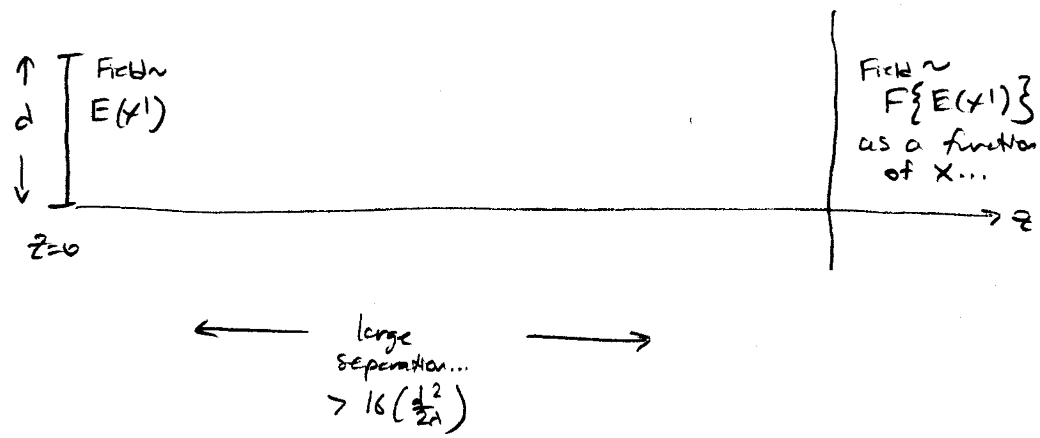
where $E(x')$ is the field distribution on the aperture. If the aperture is a slit of width 2ℓ, the limits on the integral become $\pm\ell$, and the field easily evaluates to

$$\vec{E}(x,z) = \hat{y}\;k\ell E_0\sqrt{\frac{2}{i\pi kz}}\;\underbrace{e^{ikz + ikx^2/2z}}_{\substack{e^{ikr}\text{ in}\\ \text{"on-axis"}\\ \text{approximation}}}\;\underbrace{\frac{\sin(kx\ell/z)}{(kx\ell/z)}}_{\text{sinc function}}$$

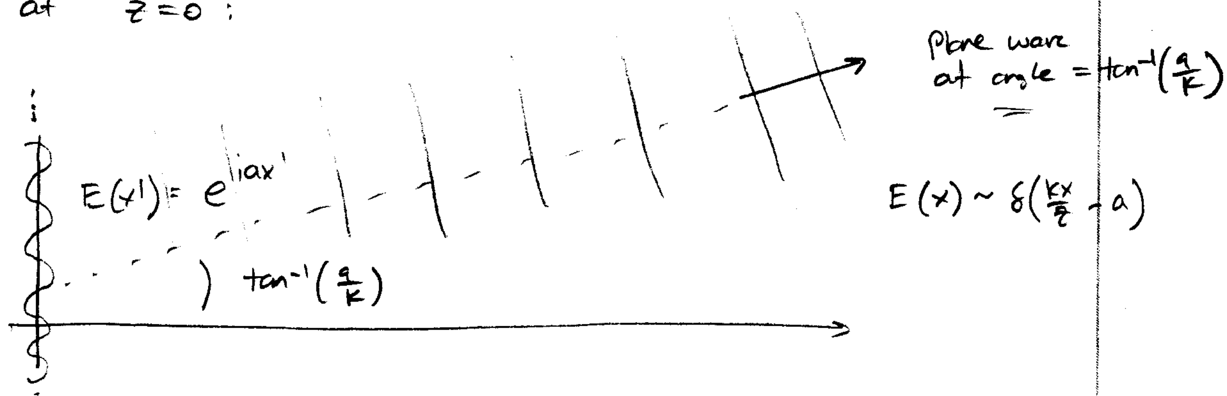
Note that the far field amplitude is proportional to the Fourier transform

$$\int_{-\infty}^{\infty} dx' \, E(x') \, e^{-i(kx/z)x'}$$

of the aperture field distribution. This is the basis for Fourier optics:



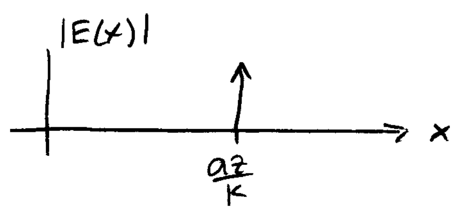
Why is this? Consider one Fourier component at the aperture at $z=0$:



The radiated field is a plane wave at an angle of

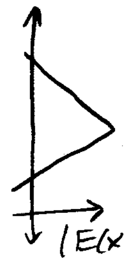
$$\tan\phi = \frac{x}{z} = \frac{az/k}{z} = \frac{a}{k}$$

As a function of x, the field distribution is a delta function!

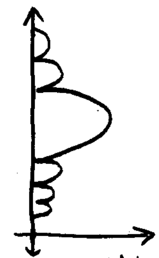


This makes sense, because we know that the Fourier transform of a complex exponential is a delta function.

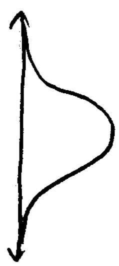
We can use this to gain intuition into the relationship between aperture distribution and far field amplitude:



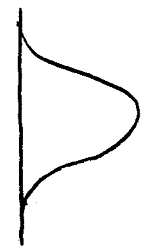
$|E(x')|\big|_{z=0} = \Lambda$



$|E(x)|\big|_{z \geq z_F} = \text{sinc}^2$

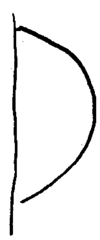


Gaussian, width $= \sigma$

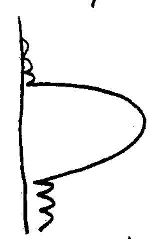


Gaussian, width $\sim 1/\sigma$

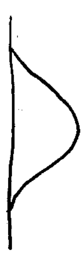
(Narrowest width of any distribution relative to aperture size...)



Blackman - Harris



Low sidelobes

 × 

$w(x') \cdot e^{iax'}$

steered beam

etc...

(Remember that the $e^{ikz + ikx^2/2z}$ is an approximation to e^{ikr} and is only valid for $x \ll z$...)