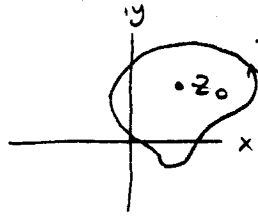
Contour integration is a tool for working integrals in the complex plane. Generally it is used to integrate functions that are singular at some point $z = z_0$ in the complex plane:



We want to find

$$\oint_C f(z)\, dz$$

To do this, we use a theorem: if $f(z)$ is analytic inside and on a contour C_2, then

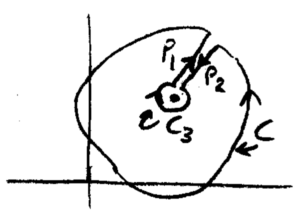
$$\int_{C_2} f(z)\, dz = 0$$

Analytic means that $f(z)$ can be expanded in a power series with no singular terms:

$$f(z) = a_0 + a_1(z - z_1) + a_2(z - z_1)^2 + \cdots$$

means that $f(z)$ is analytic at $z = z_1$.

Consider the contour C_2 defined by C and C_3:

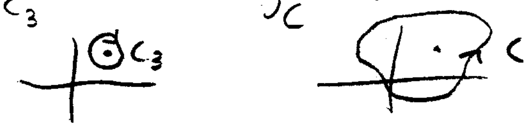


Using the theorem, $\int_{C_2} f(z)\, dz = 0$ since z_0 is not inside C_2.
But

$$\int_{C_2} f(z)\, dz = \int_{C_3} f(z)\, dz + \underbrace{\int_{P_1} f(z)\, dz + \int_{P_2} f(z)\, dz}_{\to 0 \text{ since } \int_{P_1} = -\int_{P_2}} + \int_C f(z)\, dz = 0$$

So that

$$\int_{C_3} f(z)\, dz = -\int_C f(z)\, dz$$



This reduces the integral over C to an integral over a small circle around $z = z_0$. We only need one more idea: at $z = z_0$, $f(z)$ can be expanded as a Laurent series:

$$f(z) = \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

if z_0 is a 1st order pole. The integral is now

$$\int_C f(z) = \int_{\substack{\text{circle} \\ \text{about } z_0}} f(z)\,dz$$

$$= \int f(ae^{i\phi})\,d(ae^{i\phi})$$

where $ae^{i\phi} = z - z_0$. Integrating one term of the series,

$$\int a_n (ae^{i\phi})^n\,d(ae^{i\phi}) = a_n \int a^n e^{in\phi} aie^{i\phi}\,d\phi$$

$$= \begin{cases} 2\pi i a_n & n = -1 \\ 0 & \text{otherwise} \end{cases}$$

Thus,

$$\boxed{\int f(z)\,dz = 2\pi i\, a_{-1}}$$

which is the residue theorem. a_{-1} is called the residue at z_0, and is

$$a_{-1} = \lim_{z \to z_0} (z - z_0) f(z) = \text{Res}\{z_0\}$$

If z_0 is a higher order pole, then

$$a_{-1} = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z)$$

We can use the residue theorem to work some integrals that are hard to do otherwise.

Example:

$$I = \int_0^\infty dz\, \frac{\sin z}{z}$$

$$= \int_0^\infty dz\, \frac{e^{iz}}{2iz} - \int_0^\infty dz\, \frac{e^{-iz}}{2iz}$$

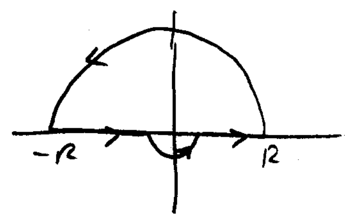
$$= \int_0^\infty dz\, \frac{e^{iz}}{2iz} - \int_0^{-\infty} d(-z)\, \frac{e^{-i(-z)}}{2i(-z)}$$

$$= \int_0^\infty dz\, \frac{e^{iz}}{2iz} + \int_{-\infty}^0 dz\, \frac{e^{iz}}{2iz}$$

$$= \int_{-\infty}^\infty dz\, \frac{e^{iz}}{2iz}$$

We can do this integral using the contour integral

$$\oint_C dz\, \frac{e^{iz}}{2iz} = 2\pi i \,\mathrm{Res}\{0\}$$



Breaking up the contour,

$$\left[\lim_{\delta\to 0}\int_\delta^\infty dz\, \frac{e^{iz}}{2iz} + \int_{-\infty}^{-\delta} dz\, \frac{e^{iz}}{2iz}\right] + \lim_{R\to\infty}\int_{C_R} dz\, \frac{e^{iz}}{2iz} + \lim_{\delta\to 0}\int_{C_\delta} dz\, \frac{e^{iz}}{2iz} = 2\pi i\,\mathrm{Res}\{0\}$$

The first integral is the value we want. The second is

$$\lim_{R\to\infty}\int_0^\pi d(Re^{i\phi})\, \frac{e^{iRe^{i\phi}}}{2iRe^{i\phi}} = \lim_{R\to\infty}\int_0^\pi d(Re^{i\phi})\, \frac{e^{iR\cos\phi}}{2iRe^{i\phi}} e^{-R\sin\phi}$$

$$= 0$$

since $\sin\phi > 0$ for $0 < \phi < \pi$, so that $e^{-R\sin\phi} \to 0$.

The third integral is

$$\lim_{\delta\to 0}\int_\pi^{2\pi} d(\delta e^{i\phi})\, \frac{e^{i\delta e^{i\phi}}}{2i\delta e^{i\phi}} = \int_\pi^{2\pi} i\,d\phi\, \frac{1}{2i} = \frac{i\pi}{2i} = \frac{\pi}{2}$$

So that

$$I + 0 + \frac{\pi}{2} = 2\pi i\,\mathrm{Res}\{0\}.$$

The residue is

$$\text{Res}\{0\} = \lim_{z \to 0} (z-0) \frac{e^{iz}}{2iz}$$

$$= \frac{1}{2i}$$

So that

$$I + \frac{\pi}{2} = 2\pi i \left(\frac{1}{2i}\right)$$

$$I = \frac{\pi}{2}$$

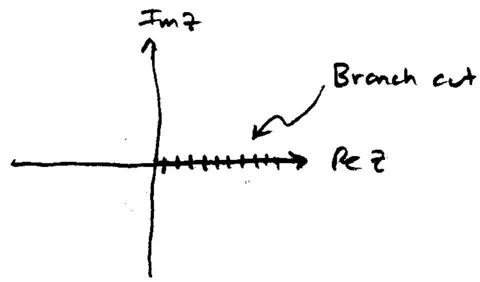
BRANCH CUTS

Some functions are multivalued, such as $f(z)=\sqrt{z}$. We need to be careful when integrating such functions in the complex plane.

The basic idea is that if we "cut" the complex plane and the contour does not cross the cut, then the function remains single valued and we can integrate the function properly.

Example: $f(z)=\sqrt{z}$

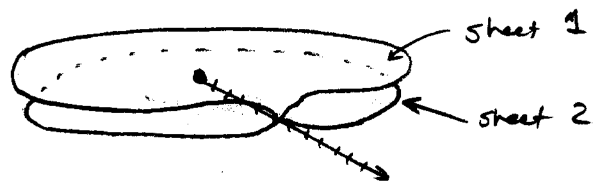
$f(z)=\sqrt{z}$ has two possible values: $\sqrt{re^{i\phi}} = \pm\sqrt{r}\, e^{i\phi/2}$. We can make it single valued with a branch cut on the $x>0$ axis:



What the cut does is create two copies of the complex plane joined along the cut. We call these "sheets" of a Riemann surface:

$$\text{Sheet 1: } re^{i\phi}, \quad 4m\pi \le \phi \le (4m+2)\pi$$

$$\text{sheet 2: } re^{i\phi}, \quad (4m+2)\pi \le \phi \le (4m+4)\pi$$



At $z=-1$, for example,

$$\text{sheet 1: } f(-1) = \sqrt{-1} = \sqrt{e^{i\pi}} = e^{i\pi/2} = i$$

$$\text{sheet 2: } f(-1) = \sqrt{-1} = \sqrt{e^{i3\pi}} = e^{i3\pi/2} = -i$$

On the Riemann surface, $f(z)$ is single valued.

To evaluate contour integrals of functions with branch cuts, we generally choose a contour that avoids the branch cut, and we end up working integrals along the cut: