For many EM problems, the desired result can be expressed in terms of a complicated integral with no closed form solution. Often, we can find an approximation as some parameter becomes large or small:

frequency $\to 0$ (static limit)
frequency $\to \infty$ (high freq. limit)
$r \to \infty$ (far field)

and so forth. One technique for doing this is asymptotic integration. There are many special cases of this approach:

Method of stationary phase:

$$\int_a^b f(x)\, e^{i\lambda g(x)}\, dx, \quad \lambda \to \infty$$

Laplace's method:

$$\int_a^b f(x)\, e^{-\lambda g(x)}\, dx, \quad \lambda \to \infty$$

Saddle-point method:

$$\int_\Gamma d\alpha\, f(\alpha)\, e^{\lambda g(\alpha)}, \quad \lambda \to \infty, \quad \Gamma \text{ is a contour in the complex plane}$$

Modified saddle-point method:
- similar to the saddle-point method, except that $f(\alpha)$ has a pole near the "saddle point"...

In all these methods, the basic idea is to find out what region dominates the value of an integral and then expand the integrand in these region, to obtain an integral that can be worked analytically.

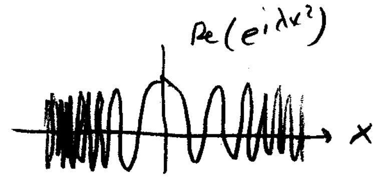
Asymptotic integration does a lot for us:
1) Simple approximate answers provide physical insight
2) Useful for design work since closed form
3) Allows numerical results to be checked

STATIONARY PHASE

We will approximate the integral

$$I(\lambda) = \int_{-1}^{1} \cos(x)\, e^{i\lambda x^2}$$

as $\lambda \to \infty$. Where $|x|$ is large, the integrand oscillates rapidly:



and the integral over adjacent periods approximately cancels, since $\cos x$ does not vary as rapidly. The main contribution to the integral comes from values of x near where $d/dx\, x^2 = 0$. This is the "critical point" of the exponent. Thus, we can expand the integrand about $x = 0$:

$$I(\lambda) \simeq \int_{-1}^{1} \cos(0)\, e^{i\lambda x^2}$$

$$= \int_{-1}^{1} e^{i\lambda x^2}$$

$$\simeq \int_{-\infty}^{\infty} e^{i\lambda x^2}$$

(Since the integrand is rapidly oscillating for $|x| > 1$ the integral over $(-\infty, -1)$ and $(1, \infty)$ is small)

$$= \sqrt{\frac{i\pi}{\lambda}}$$

For $\lambda = 5$,

$$I(\lambda) \cong .483 + .527i$$

$$\sqrt{\frac{i\pi}{5}} \cong .560 + .560i$$

For $\lambda = 100$,

$$I(\lambda) \cong .1229 + .1204i$$

$$\sqrt{\frac{i\pi}{100}} \cong .1253 + i.1253i$$

and the approximation clearly becomes better as λ grows.

LAPLACE'S METHOD

Laplace's method is similar to stationary phase, except that the integrand becomes small where $g(x)$ is large:

$$\int_a^b f(x)\, \underbrace{e^{-\lambda g(x)}}_{\substack{\text{this is small} \\ \text{if } \lambda g \text{ is large}}}\, dx \quad , \quad \lambda \to \infty$$

Where is the integrand largest? If $g(x) > 0$, it occurs at point(s) where

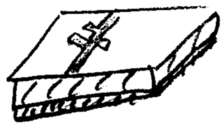
$$g'(x_0) = 0$$

After finding these critical point(s), we expand $g(x)$ and $f(x)$ about x_0 and evaluate the resulting integral.

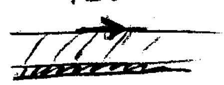
The method of steepest descents has two important applications:

1) Finding closed form approximations to integrals

2) Numerical evaluation of integrals — e.g., in a simulation software package for microstrip circuits:



→ Need Green's function HED



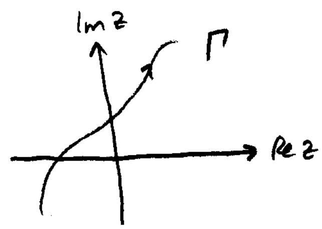
→ Dipole on layered medium: $\int dk_\rho \, f(k_\rho) e^{i\lambda g(k_\rho)}$

⇓

Sample integrand on <u>SDP</u>

The goal is to work integrals of the form

$$I(\lambda) = \int_\Gamma dz \, g(z) e^{\lambda f(z)}$$



(Note: Γ is often the real axis...)

We use the analyticity of the integrand to move the contour Γ to a new contour that passes through a critical point, which for the SD method is also called a saddle point.

<u>Note</u>: If the contour crosses a <u>pole</u> or must go around a <u>branch cut</u> as it is deformed from Γ to the new contour, the integral's value changes — but this change can be found using residue theory or by working integrals around the branch cut.

A critical point is defined by

$$\boxed{\frac{d f(z)}{dz} = 0} \rightarrow \text{critical point } z_0$$

Any analytic function $f(z) = u(x,y) + i v(x,y)$ must satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

This is easy to see from the definition of analyticity:

$$f'(z) \text{ exists} \rightarrow f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

Let $\Delta z = \Delta x$. $f'(z) = \lim_{\Delta x \to 0} \left[\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} \right]$

$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

Let $\Delta z = i \Delta y$. $f'(z) = \lim_{\Delta y \to 0} \left[\frac{u(x, y+\Delta y) - u(x,y)}{i \Delta y} + i \frac{v(x, y+\Delta y) - v(x,y)}{i \Delta y} \right]$

$$= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Thus, $\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \Rightarrow$ Cauchy-Riemann eqs...

We can derive Laplace's eqn. from these equations:

$$0 = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right)$$

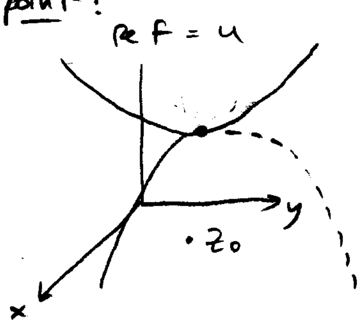
$$= \frac{\partial^2 u}{\partial x^2} - \frac{\partial}{\partial y} \frac{\partial v}{\partial x}$$

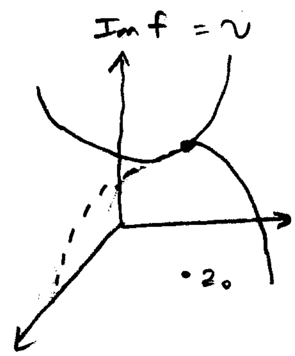
$$= \underline{\underline{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}}}, \quad \text{also } \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \Rightarrow \underline{\underline{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0}}$$

At the critical point $\overset{f}{\cancel{g}}{}'(z_0) = 0$, $\frac{\partial \overset{f}{\cancel{g}}}{\partial x} = \frac{\partial \overset{f}{\cancel{g}}}{\partial y} = 0$. But Laplace's eqn. says that

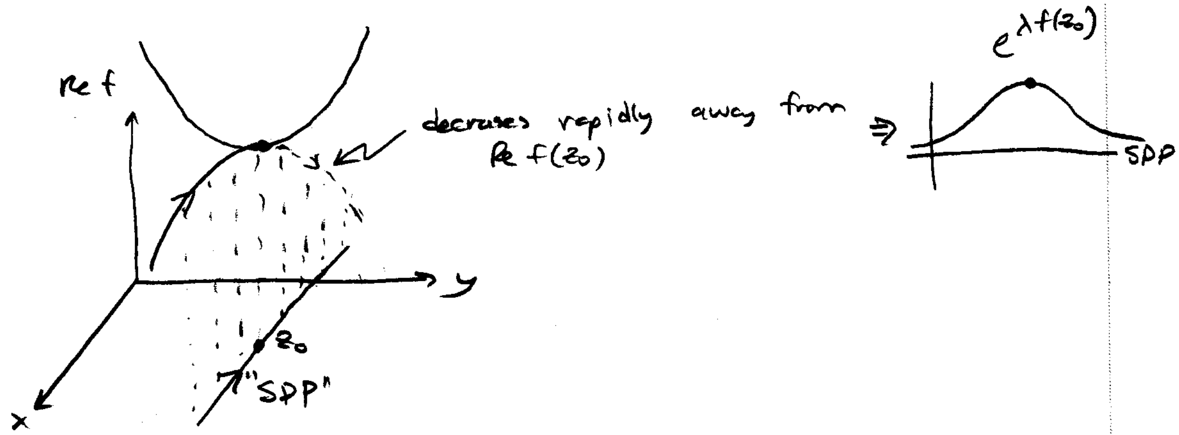
$$\frac{\partial^2 f}{\partial x^2} = -\frac{\partial^2 f}{\partial y^2}$$

So that z_0 is neither a maximum nor a minimum, but a <u>saddle point</u>:





The new contour through the saddle point z_0 is chosen so that the real part of $f(z)$ decreases as fast as possible:



This is the steepest descent path (SPP).

To make sure this works, we need to be sure that the imaginary part of $f(z)$ is not rapidly oscillating. Let's check this:

The direction of most rapid change of $Re\{f\} = u(x,y)$ is

$$\nabla u = \hat{x}\frac{\partial u}{\partial x} + \hat{y}\frac{\partial u}{\partial y}$$

The change in $Im\{f\} = v(x,y)$ in this direction is

$$(\nabla v)\cdot\nabla u = \left(\hat{x}\frac{\partial v}{\partial x} + \hat{y}\frac{\partial v}{\partial y}\right)\cdot\left(\hat{x}\frac{\partial u}{\partial x} + \hat{y}\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y}$$

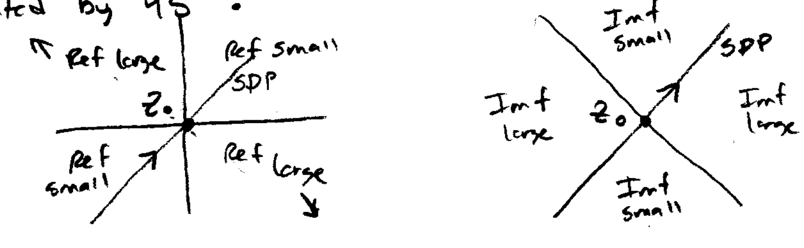
$$= \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial v}{\partial y}\left(-\frac{\partial v}{\partial x}\right)$$

$$= 0 \quad !!$$

* Thus, $Im\{f\}$ is not changing at z_0, and the <u>steepest descent</u> path (for $Re\{f\}$) is also the path of stati<u>onary phase</u> (for $Im\{f\}$)! *

This is why the SD method works.

In the picture, this means that the saddles for Ref and Imf are rotated by 45°:

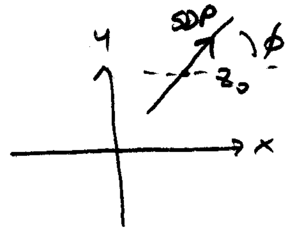


Now let's use all this to work the integral. Near z_0,

$$f(z) \cong f(z_0) + \cancel{f'(z_0)}^{\,0} + \frac{(z-z_0)^2}{2} f''(z_0)$$

The SDP is of the form

$$z = z_0 + e^{i\phi} t$$



The integral is

$$I(\lambda) = \int_{SDP} dz\, g(z)\, e^{\lambda f(z)} \qquad \left(+ \text{Residues if contour crosses over poles} \ldots\right)$$

$$\cong \int_{\infty}^{-\infty} d(z_0 + e^{i\phi} t)\, g(z_0)\, e^{\lambda \left(f(z_0) + \frac{(e^{i\phi} t)^2}{2} f''(z_0)\right)}$$

$$= g(z_0) e^{\lambda f(z_0)} \int_{-\infty}^{\infty} e^{i\phi} dt\, e^{(\lambda f''(z_0) e^{i2\phi}/2) t^2}$$

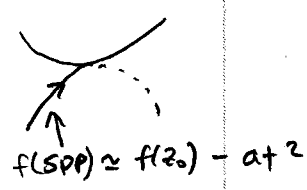
We choose the SDP so that

$$\mathrm{Im}\{f(z)\} = \mathrm{Im}\{f(z_0)\} = \underline{\text{constant}}$$

On this path,

$$e^{\underbrace{\lambda f''(z_0) e^{i2\phi}/2}_{\text{real, negative}} t^2} = e^{-\lambda |f''(0)| t^2}$$

Since $f(z) \cong f(z_0) - \frac{t^2}{2} |f''(z_0)|$ on the SDP.



We can now work the integral:

$$I(\lambda) \cong g(z_0) e^{\lambda f(z_0)} e^{i\phi} \int_{-\infty}^{\infty} dt\, e^{+\lambda f''(z_0) e^{i2\phi} t^2/2}$$

$$= g(z_0) e^{\lambda f(z_0)} e^{i\phi} \sqrt{\frac{-2\pi}{\lambda e^{i2\phi} f''(z_0)}} \quad \text{using } \int_{-\infty}^{\infty} dt\, e^{-at^2} = \sqrt{\frac{\pi}{a}}$$

Cancelling the factors of $e^{i\phi}$ leads to

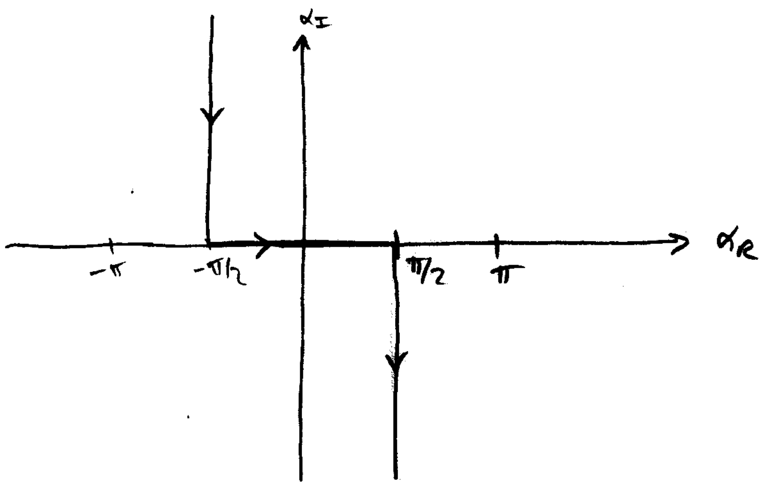
$$\underline{\underline{I(\lambda) \cong g(z_0) e^{\lambda f(z_0)} \sqrt{\frac{-2\pi}{\lambda f''(z_0)}}}}$$

Saddle-point method:

Let's approximate the Hankel function $H_\nu^{(1)}(\xi)$ as $|\xi| \to \infty$. The Hankel function has an integral representation,

$$H_\nu^{(1)}(\xi) = \frac{1}{\pi} \int_{\Gamma_1} d\alpha \; e^{i(\xi \cos\alpha + \nu\alpha - \nu\pi/2)}$$

where Γ is the path



The saddle point method is similar to the method of stationary phase. The integrand is rapidly oscillating for large ξ, except where

$$\frac{d}{d\alpha}(\xi \cos\alpha) = 0$$

Such points are called "saddle points," and are analogous to critical points of the method of stationary phase. For this example, the saddle points are $\alpha = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$

The next step is to deform the path Γ_1 to a new path which passes through a saddle point. We can do this using Cauchy's theorem, since the integrand has no singularities. We want the path to be such that the integral is determined by as small a region as possible near the saddle point. This is done by finding the path for which the imaginary part of the exponent is constant:

$$\mathrm{Im}\{\underbrace{g(\alpha)}_{\text{exponent}}\} = \mathrm{Im}\{g(\underset{\substack{\uparrow \\ \text{saddle point}}}{\alpha_0})\}$$

So that

$$\mathrm{Im}\{i\xi\cos\alpha\} = \mathrm{Im}\{i\xi\cos(0)\}$$

$$\mathrm{Im}\{i\cos\alpha\} = 1 \qquad (\text{since } \xi \text{ is real})$$

$$= \operatorname{Im}\{ i \cos(\alpha_R + i\alpha_I) \}$$

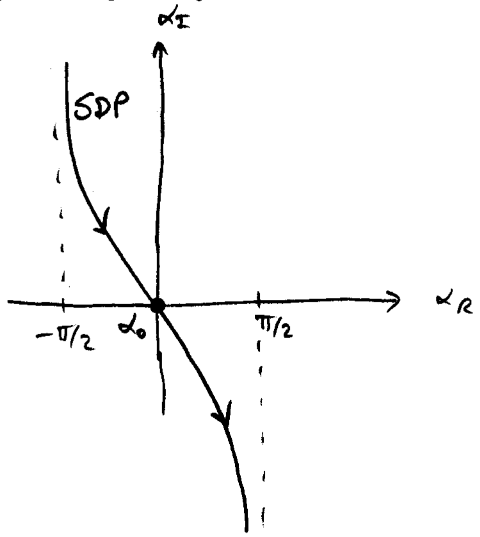
$$= \operatorname{Im}\{ i (\cos\alpha_R \cos(i\alpha_I) - \sin\alpha_R \sin(i\alpha_I)) \}$$

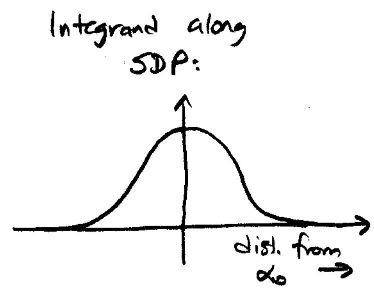
$$= \operatorname{Im}\{ i (\cos\alpha_R \cosh\alpha_I - i \sin\alpha_R \sinh\alpha_I) \}$$

$$= \operatorname{Im}\{ i \cos\alpha_R \cosh\alpha_I + \sin\alpha_R \sinh\alpha_I \} = 1$$

$$\underline{\cos\alpha_R \cosh\alpha_I = 1}$$

Along the path defined by this result, the imaginary part of the exponent is constant, and the real part is maximum at $\alpha_0 = 0$ (the saddle point) and decreases rapidly as α moves away from $\alpha_0 = 0$. This is the "steepest descent path" (SDP):





We now integrate over the SDP:

$$\frac{1}{\pi} \int_{SDP} d\alpha\, e^{i(\xi\cos\alpha + \nu\alpha - \nu\pi/2)}$$

$$= \frac{1}{\pi} \int_{SDP} d\alpha\, e^{i\xi\cos\alpha_R\cosh\alpha_I + \xi\sin\alpha_R\sinh\alpha_I}\, e^{i(\nu\alpha - \nu\pi/2)}$$

Exanding the integrand arand $\alpha = \alpha_0$,

$$H_\nu^{(1)}(\xi) \simeq \frac{1}{\pi} e^{-i\nu\pi/2} \int_{SDP} d\alpha\, e^{i\xi + \xi\alpha_R\alpha_I}$$

Now, we change variables to a parameter for the path. Near $\alpha_0 = 0$, the SPP is defined by

$$1 = \cos\alpha_R\cosh\alpha_I \simeq \left(1 - \frac{\alpha_R^2}{2}\right)\left(1 + \frac{\alpha_I^2}{2}\right) \simeq 1 - \frac{\alpha_R^2}{2} + \frac{\alpha_I^2}{2}$$

or $\alpha_R^2 \simeq \alpha_I^2$. From the picture, we see that the correct sign is

$$\alpha_R \simeq -\alpha_I \,.$$

Let the parameter $s = \alpha_R$. Then the SPP is

$$\alpha(s) \simeq s - is$$

The integral becomes

$$H_\nu^{(1)}(\xi) \simeq \frac{1}{\pi} e^{-i\nu\pi/2} e^{i\xi} \int_{-\delta}^{\delta} d\alpha(s)\, e^{\xi s(-s)}$$

$$= \frac{1}{\pi} e^{i(\xi-\nu\pi/2)} \int_{-\delta}^{\delta} \frac{\partial\alpha}{\partial s}\, ds\, e^{-\xi s^2}$$

$$= \frac{1}{\pi} e^{i(\xi-\nu\pi/2)} \int_{-\infty}^{\infty} (1-i)\, ds\, e^{-\xi s^2}$$

$$= \frac{1}{\pi} e^{i(\xi-\nu\pi/2)} \sqrt{2}\, e^{-i\pi/4} \int_{-\infty}^{\infty} ds\, e^{-\xi s^2}$$

$$= \frac{\sqrt{2}}{\pi} e^{i(\xi-\nu\pi/2-\pi/4)} \sqrt{\frac{\pi}{\xi}}$$

$$= \sqrt{\frac{2}{\pi\xi}}\, \underline{\underline{e^{i(\xi-\nu\pi-\pi/4)}}}$$

This is the large-argument approximation to $H_\nu^{(1)}(\xi)$.