Array theory is a sequence of increasing generality:

simplest array { (1D) Linear array, equidistant, equal excitation

↓

change driving current {

Progressive phase - $e^{in\alpha}$ excitation ⇒ beam steering

↓

Nonuniform excitation
- binomial ⇒ sinc^2 pattern
- Dolph-Chebyshev ⇒ equal ripple - lowest sidelobes
- z-transform method
- Pattern synthesis
  - Lagrange interpolation

}

↓

change location {

Irregular spacing
- thinned arrays
- random arrays ⇒ no grating lobes

↓

}

2D arrays
- active antennas
- 2D beam steering

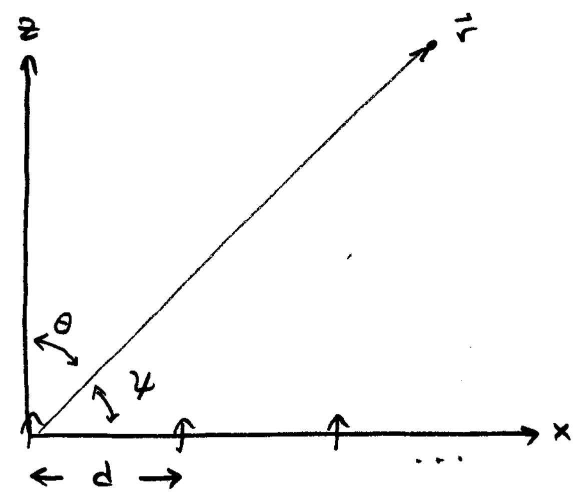
⋮

Recall from 360 that the far field radiation pattern of a linear array of antenna elements is

$$S(\theta) = \underbrace{S_{\text{element}}(\theta)}_{\substack{\text{Individual}\\ \text{element}\\ \text{pattern}}} \; \underbrace{F_{\text{array}}(\theta)}_{\text{Array factor}}$$

$$F_{\text{array}}(u) = \left| \frac{\sin(Nu/2)}{\sin(u/2)} \right| = \text{periodic sinc or Dirichlet function...}$$

$$u = kd\cos\psi - \alpha$$

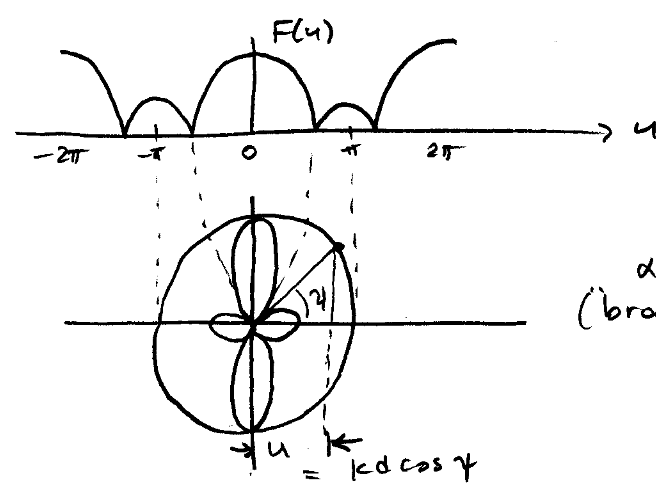


Element: $J_n(\vec{r}) = \hat{z}\, I\ell \underbrace{e^{in\alpha}}_{\substack{\text{Driving}\\ \text{phase}}} \underbrace{\delta(x-nd)}_{\text{location}} \delta(y)\delta(z)$

Example: $N=3$, $\alpha=0$, $\hat{y}$-directed dipoles, $d = \lambda/2$

$kd = \frac{2\pi}{\lambda}\cdot\frac{\lambda}{2} = \pi$

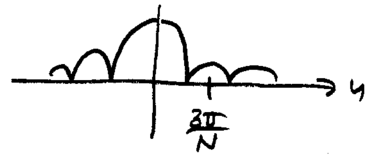
"Visible window" method



$\alpha=0$: ("broadside")

First Sidelobe

$$F_{max} = F(0) = \left|\frac{\sin(Nu/2)}{\sin(u/2)}\right| = N$$



$$F_{1st\ sidelobe} = F(3\pi/N) = \left|\frac{\sin(3\pi/2)}{\sin(3\pi/2N)}\right|$$

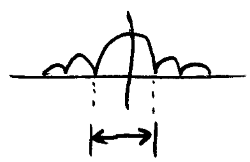
$$= \frac{1}{|\sin(3\pi/2N)|}$$

$$\simeq \frac{2N}{3\pi}, \quad N \to \infty$$

$$\Rightarrow \text{First sidelobe is} \approx \frac{(2N/3\pi)}{N} = \frac{2}{3\pi} = \boxed{-13.5\text{ dB}} \text{ lower...}$$

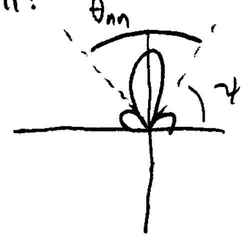
Beamwidth

Broadside:



$u = 2\pi/N$

Null-Null:



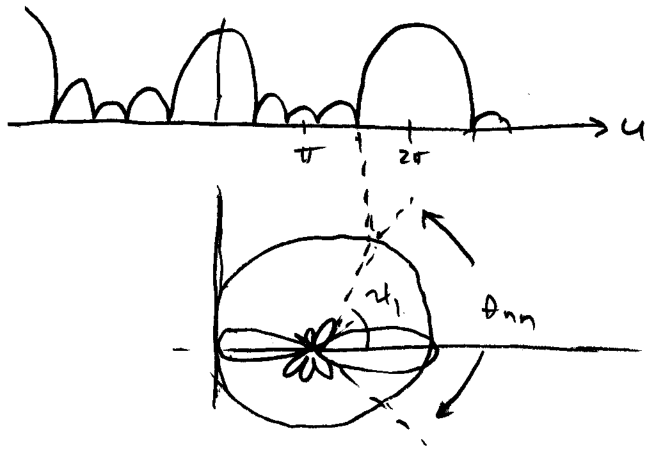
$$kd\cos\psi_1 = \frac{2\pi}{N}$$

$$kd\sin(\theta_1) = 2\pi/N$$

$$\theta_1 = \sin^{-1}\frac{2\pi}{Nkd} \simeq \frac{2\pi}{Nkd}, \quad N \to \infty$$

$$\boxed{\theta_{n-n} = \frac{4\pi}{Nkd}}$$

End fire:
($\alpha = \pi$)



$$u_1 = 2\pi - \frac{2\pi}{N} = kd\cos\psi_1 + kd$$

$$kd - \frac{2\pi}{N} = kd\cos\psi_1$$

$$\cos\psi_1 = 1 - \frac{2\pi}{Nkd}$$

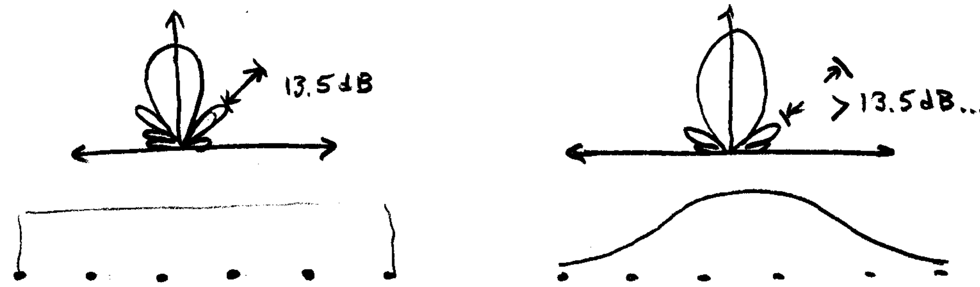
$$1 - \frac{\psi_1^2}{2} \simeq 1 - \frac{2\pi}{Nkd}, \quad N \to \infty$$

$$\psi_1 = \sqrt{\frac{4\pi}{Nkd}}$$

$$\boxed{\theta_{nn} = 2\sqrt{\frac{4\pi}{Nkd}}} \quad \Big\} \text{ larger!}$$

<u>Nonuniform excitation</u>

Previously, we have considered only arrays with equal-amplitude driving currents. We can gain additional control by changing the amplitude. This could be used, for example, to reduce sidelobes:



Equal amplitude   Tapered

A simple possibility is a triangle weighting:

$$F(u) = 1 + 2e^{-iu} + \cdots + Ne^{-i(N-1)u} + \cdots + 2e^{i(2N-3)u} + e^{-i(2N-1)}$$

Weighting = 

2N-1 elements

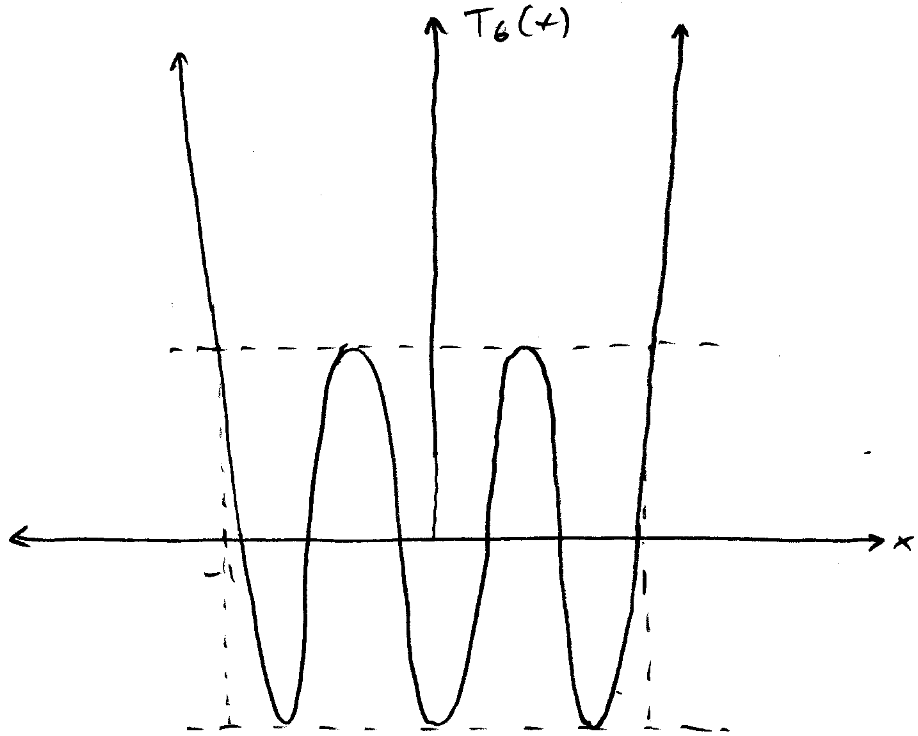
$\Rightarrow F(u) \ni \mathrm{sinc}^2 \Rightarrow 1^{st}$ sidelobe $\cong 2 \cdot 13.5 = \underline{\underline{27\ dB}}$

There are a number of different methods for choosing the weights of a nonuniformly excited array. Each approach has a goal of some kind, which might include

1. Narrow main beam
2. low sidelobes
3. Some set array pattern
4. Stable beam when electronically steered
5. Flat main beam 

and so forth.

Chebyshev polynomials oscillate between $[1,-1]$ from $-1 \le x \le 1$, and grow in magnitude outside this range:

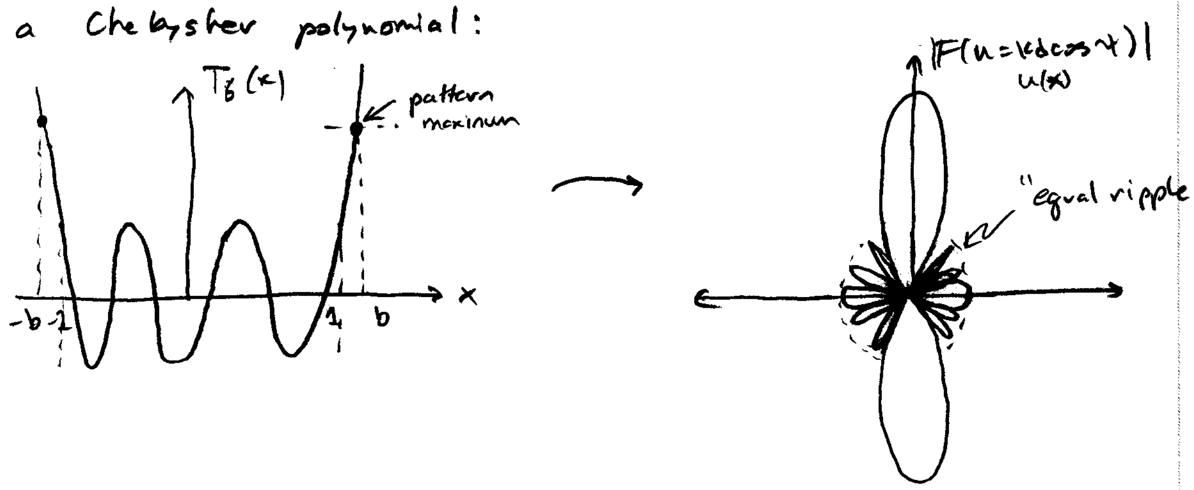


where $T_n(x) \equiv \cos(n\cos^{-1}(x))$. The property of the maxima all at 1 or -1 is called "equal ripple" for obvious reasons. These polynomials have important properties and are used in many applications.

The basic idea for use in array synthesis is to transform the u variable to a range such that

$$-b \le x(u) \le b$$

where $b > 1$. Then we set the array factor $F(u)$ equal to a Chebyshev polynomial:



It can be shown that this type array has the smallest sidelobe level for a given main beam width.

The first few Chebyshev polynomials are

$$T_n(x) = \cos(n\cos^{-1}x)$$

$$\left.\begin{array}{l} T_0(x) = 1 \\ T_1(x) = x \\ T_2(x) = 2x^2 - 1 \\ T_3(x) = 4x^3 - 3x \end{array}\right\} \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

In the array factor $F(u)$, we let

$$x = b\cos\frac{u}{2} \qquad \text{(Dolph transformation)}$$

and write F in the form

$$F(u) = \begin{cases} 2\sum_{m=1}^{N/2} a_m \cos\left(\frac{2m-1}{2}\right)u & N = \text{even} \\ a_0 + 2\sum_{m=1}^{(N-1)/2} a_m \cos(mu) & N = \text{odd} \end{cases}$$

by combining symmetric terms of $F(u)$.
Equating

$$F(u) = T_{N-1}(x)$$

gives

$$2\sum_{m=1}^{N/2} a_m \underbrace{\cos\left[\left(\frac{2m-1}{2}\right)2\cos^{-1}(x/b)\right]}_{T_{2m-1}} = T_{N-1}(x) \qquad (N \text{ even})$$

$$2\sum_{1}^{N/2} a_m T_{2m-1}(x/b) = T_{N-1}(x)$$

with a similar expression for N odd.
By expanding both sides as polynomials, we can find the a_m.

Example $N=5$, $d=\lambda/2$, $\alpha=0$.

From above, $F(u) = T_4(x)$, $x = b\cos(u/2)$. For N odd,

$$a_0 + 2\sum_{m=1}^{(N-1)/2} a_m T_{2m}(x/b) = T_4(x)$$

Expanding the sum,

$$a_0 + 2a_1 T_2(x/b) + 2a_2 T_4(x/b) = T_4(x)$$

$$a_0 + 2a_1\left(2(x/b)^2 - 1\right) + 2a_2\left(8(x/b)^4 - 8(x/b)^2 + 1\right) = 8x^4 - 8x^2 + 1$$

$$x^4 \rightarrow \quad 2a_2 8(1/b)^4 = 8 \quad \rightarrow \quad a_2 = \underline{b^4/2} \quad ✓$$

$$x^2 \rightarrow \quad 4a_1(1/b)^2 - 16a_2(1/b)^2 = -8$$

$$4a_1/b^2 - 8b^4/b^2 = -8$$

$$4a_1/b^2 = 8b^2 - 8$$

$$a_1 = \frac{8b^2 - 8}{4} b^2 = +2\underline{b^4 - 2b^2} \quad ✓$$

$$x^0 \rightarrow \quad a_0 - 2a_1 + 2a_2 = 1$$

$$a_0 = 2(+2b^4 - 2b^2) - 2b^4/2 + 1$$

$$= +4b^4 - 4b^2 - b^4 + 1$$

$$= \underline{3b^4 - 4b^2 + 1} \quad ✓$$

The zeros of $T_n(x)$ occur at $x = \cos\left(\frac{(2p-1)\pi}{2n}\right)$, $p = 1, 2, \ldots n$.
The first null is at

$$x = \cos\left(\frac{\pi}{2(N-1)}\right) = b\cos(u/2) \rightarrow \text{solve for } b\ldots$$

The maximum is at $u=0$, where $F(0) = T_{N-1}(b)$. Or, we can fix the sidelobe level and solve for the null-null beamwidth.