For a uniaxial medium,

$$\bar{\bar{\epsilon}} = \begin{bmatrix} \epsilon & & \\ & \epsilon & \\ & & \epsilon_z \end{bmatrix}, \qquad \bar{\bar{\kappa}} = \begin{bmatrix} 1/\epsilon & & \\ & 1/\epsilon & \\ & & 1/\epsilon_z \end{bmatrix} = \begin{bmatrix} \kappa & & \\ & \kappa & \\ & & \kappa_z \end{bmatrix}$$

Suppose that $\vec{k}$ is in the $y-z$ plane, so that

$$\hat{k} = \hat{y}\sin\theta + \hat{z}\cos\theta$$

Then

$$\hat{k}\hat{k}^T = \begin{bmatrix} 0 \\ \sin\theta \\ \cos\theta \end{bmatrix} \begin{bmatrix} 0 & \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sin^2\theta & \sin\theta\cos\theta \\ 0 & \sin\theta\cos\theta & \cos^2\theta \end{bmatrix}$$

and

$$v\left[\bar{\bar{I}} - \hat{k}\hat{k}^T\right]\bar{\bar{\kappa}} = v\left\{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sin^2\theta & \sin\theta\cos\theta \\ 0 & \sin\theta\cos\theta & \cos^2\theta \end{bmatrix}\right\}\begin{bmatrix} \kappa & & \\ & \kappa & \\ & & \kappa_z \end{bmatrix}$$

$$= v\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos^2\theta & -\sin\theta\cos\theta \\ 0 & -\sin\theta\cos\theta & \sin^2\theta \end{bmatrix}\begin{bmatrix} \kappa & & \\ & \kappa & \\ & & \kappa_z \end{bmatrix}$$

$$= \begin{bmatrix} v\kappa & 0 & 0 \\ 0 & v\kappa\cos^2\theta & -v\kappa_z\sin\theta\cos\theta \\ 0 & -v\kappa\sin\theta\cos\theta & v\kappa_z\sin^2\theta \end{bmatrix}$$

The eigenvalues and eigenvectors of this matrix are

$$\lambda_1 = v\kappa, \qquad \text{eigenvector} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_2 = v\kappa\cos^2\theta + v\kappa_z\sin^2\theta, \qquad \text{eigenvector} = \begin{bmatrix} 0 \\ -\cos\theta \\ \sin\theta \end{bmatrix}$$

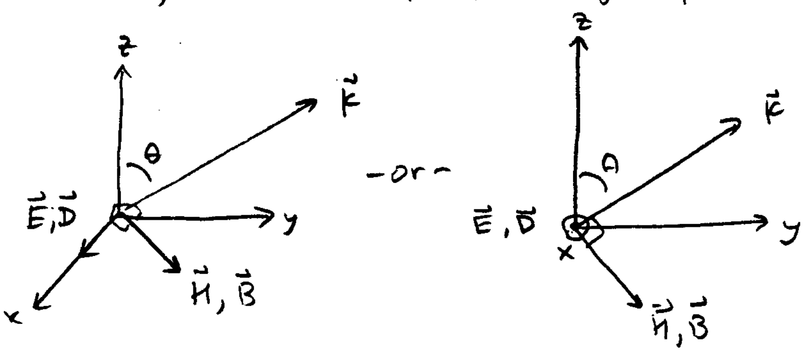
$$\lambda_3 = 0, \qquad \text{eigenvector} = \begin{bmatrix} 0 \\ v\kappa_z\sin\theta \\ v\kappa\cos\theta \end{bmatrix}$$

All that remains is to physically interpret these solutions.

<u>Case I</u> ($\lambda_1 = \nu\kappa$)

$$\lambda_1 = \nu\kappa = \frac{\omega^2}{k^2} = \frac{1}{\mu\epsilon} \Rightarrow k = \omega\sqrt{\mu\epsilon}$$

Polarization: $\vec{D} = D_0\hat{x}$, $\vec{E} = E_0\hat{x}$, $\vec{H}, \vec{B}$ in y-z plane



$\vec{S} = \vec{E} \times \vec{H}^*$ is in the $\vec{k}$ direction

$\Rightarrow$ $\boxed{\text{"Ordinary wave"}}$

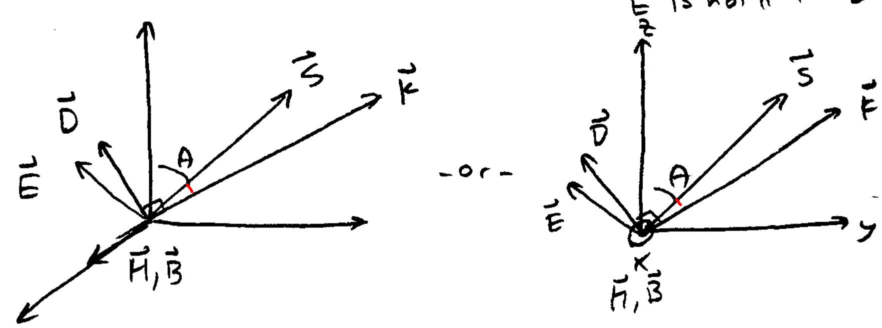
<u>Case II</u>

$$\lambda_1 = \nu\kappa\cos^2\theta + \nu\kappa_z\sin^2\theta \Rightarrow \frac{\omega^2}{k^2} = \frac{\cos^2\theta}{\epsilon\mu} + \frac{\sin^2\theta}{\epsilon_z\mu}$$

$$k = \omega\sqrt{\frac{1}{\frac{\cos^2\theta}{\epsilon\mu} + \frac{\sin^2\theta}{\epsilon_z\mu}}}$$

Polarization: $\vec{D} = D_0(-\hat{y}\cos\theta + \hat{z}\sin\theta)$, $\vec{E} = D_0\left(-\hat{y}\,\overset{\kappa}{\cancel{\nu}}\cos\theta + \hat{z}\,\overset{\kappa_z}{\cancel{\nu_z}}\sin\theta\right)$

$\vec{E}$ is not $\parallel$ to $\vec{D}$



$\vec{S} = \vec{E} \times \vec{H}^*$ is not $\parallel$ to $\vec{k}$

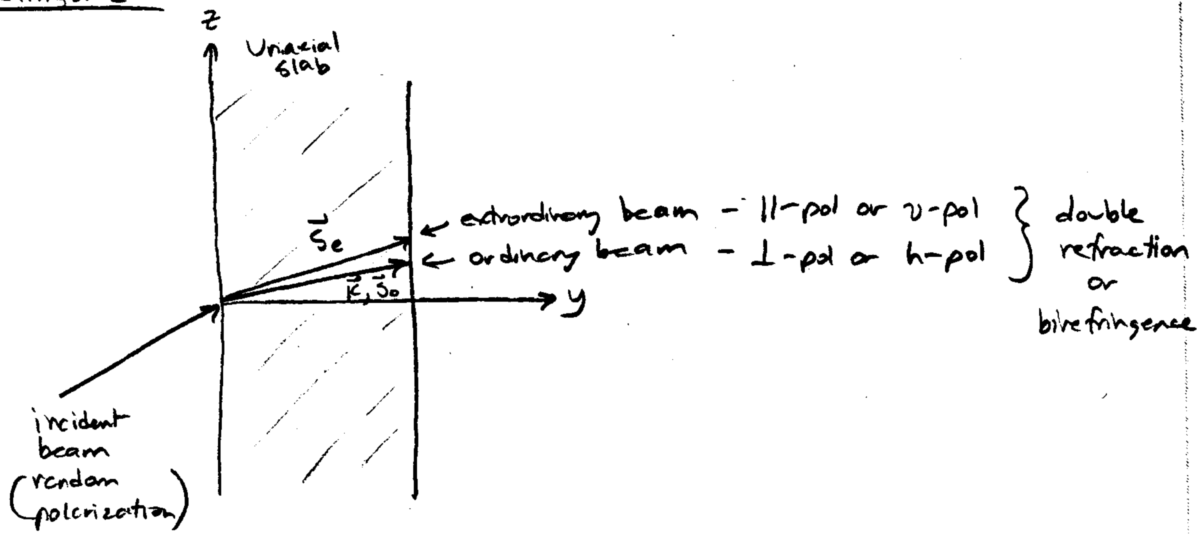
$\Rightarrow$ $\boxed{\text{"Extraordinary wave"}}$

<u>Case III</u>

$\lambda_1 = 0 \Rightarrow \omega = 0$, $\vec{E} = (\hat{y}\sin\theta + \hat{z}\cos\theta) \parallel \vec{k}$ → longitudinal wave

- does not propagate

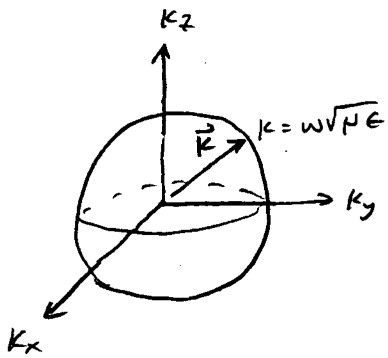
Birefringence



$\vec{k}$ surface

In free space, $k = \omega\sqrt{\mu\epsilon}$ describes a sphere in "k-space":

Free space:

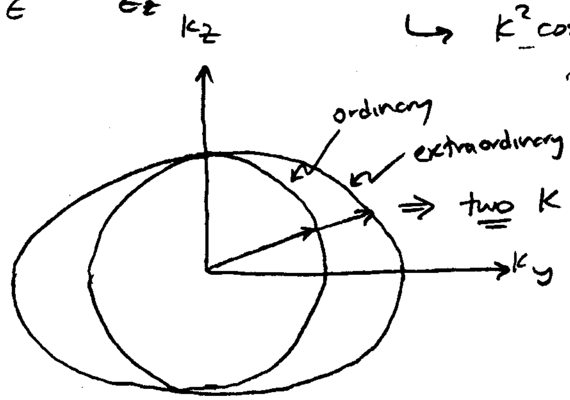


In a uniaxial medium, the sphere splits into two surfaces:

$k = \omega\sqrt{\mu\epsilon}$ (ordinary wave) = sphere → $k^2\sin^2\theta + k^2\cos^2\theta = \omega\sqrt{\mu\epsilon}$

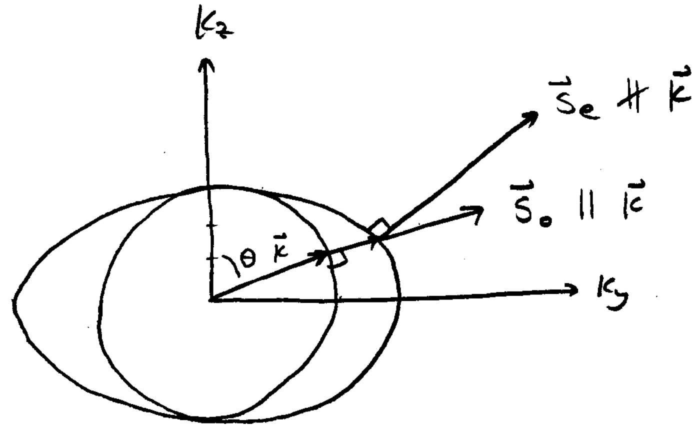
$k = \omega\sqrt{\mu}\sqrt{\dfrac{1}{\dfrac{\cos^2\theta}{\epsilon} + \dfrac{\sin^2\theta}{\epsilon_z}}}$ (extraordinary wave) = ellipsoid

↳ $k^2\cos^2\theta + \dfrac{\epsilon}{\epsilon_z}k^2\sin^2\theta \quad \omega\sqrt{\mu\epsilon}$



⇒ <u>two</u> k values for each direction $\hat{k}$

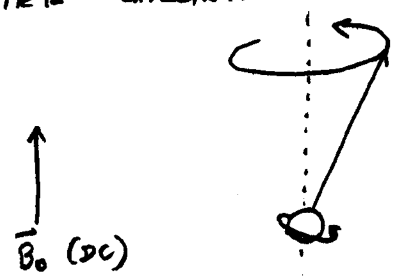
It can be shown that $\vec{S}$ is normal to the $\vec{k}$ surface:



There are two types of gyrotropic material. The first is gyromagnetic:

$$\bar{\bar{\mu}} = \begin{bmatrix} \mu & i\mu_g & 0 \\ -i\mu_g & \mu & 0 \\ 0 & 0 & \mu_z \end{bmatrix}$$

An example of this type of material is a ferrite in a DC bias magnetic field. The bias field causes the spins of each atom to line up. If the field is strong enough, the spins are all lined up, or saturated. The spin directions precess around the bias field direction:



Now, if an AC magnetic field is applied, the motion of the spin direction is perturbed. The magnetic moments of the spins then produce an additional contribution to the magnetic flux, according to the constitutive relationship above.

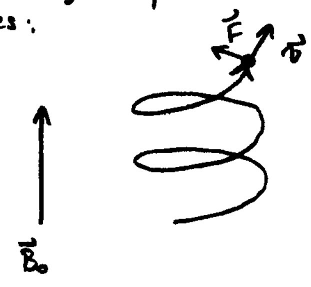
The second type is gyroelectric:

$$\bar{\bar{\epsilon}} = \begin{bmatrix} \epsilon & i\epsilon_g & 0 \\ -i\epsilon_g & \epsilon & 0 \\ 0 & 0 & \epsilon_z \end{bmatrix}$$

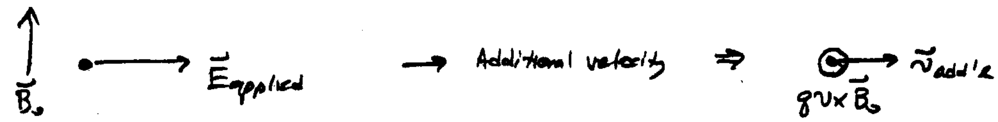
A plasma in a DC bias magnetic field is gyroelectric. Because of the Lorentz force law,

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

the force on each charge is orthogonal to its direction of motion. This causes the charged particles in the plasma to spiral around the $\vec{B}$ field lines:



An AC applied electric field perturbs the spiral motion of the charges, which in turn produces electric flux, leading to the gyroelectric constitutive relations:



The additional forces lead to a motion of charges in the direction of $E_{applied}$ and orthogonal to it:

$$\begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix} = \begin{bmatrix} \alpha & -\gamma & 0 \\ \gamma & \alpha & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \bar{\bar{\sigma}}\,\vec{E}$$

Using Ampere's law, we can incorporate this into $\bar{\bar{\epsilon}}$:

$$\nabla\times\vec{H} = -i\omega\epsilon_0\vec{E} + \bar{\bar{\sigma}}\vec{E}$$

$$= -i\omega\underbrace{\left(\epsilon_0 + \frac{\bar{\bar{\sigma}}}{-i\omega}\right)}_{\bar{\bar{\epsilon}}}\vec{E}$$

where

$$\bar{\bar{\epsilon}} = \begin{bmatrix} \epsilon & -i\epsilon_g & 0 \\ +i\epsilon_g & \epsilon & 0 \\ 0 & 0 & \epsilon_z \end{bmatrix} \qquad i\epsilon_g = -\frac{i}{\omega}-\gamma$$

and $\epsilon = \epsilon_0 + \alpha/j\omega$, $\epsilon_g = \gamma/\omega$.

For a gyroelectric medium, the impermittivity matrix is of the form

$$\bar{\bar{\kappa}} = \begin{bmatrix} \kappa & i\kappa_g & 0 \\ -i\kappa_g & \kappa & 0 \\ 0 & 0 & \kappa_z \end{bmatrix}$$

For a plane wave, Maxwell's equations reduce to

$$(\bar{\bar{I}} - \hat{k}\hat{k}^T)\bar{\bar{\kappa}}\vec{D} - \frac{\omega^2\mu}{k^2}\bar{D} = 0$$

The solutions to this equation are the eigenvalues and eigenvectors of the matrix

$$(\bar{\bar{I}} - \hat{k}\hat{k}^T)\bar{\bar{\kappa}}$$

To simplify the treatment, assume the plane wave is propagating in the z direction, so that $\vec{k} = k\hat{z}$. Then $\hat{k} = [0\ 0\ 1]^T$, and

$$(\bar{\bar{I}} - \hat{k}\hat{k}^T)\bar{\bar{\kappa}} = \left\{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}[0\,0\,1]\right\}\bar{\bar{\kappa}}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}\bar{\bar{\kappa}}$$

$$= \begin{bmatrix} \kappa & i\kappa_g & 0 \\ -i\kappa_g & \kappa & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Clearly, one eigenvalue is $\lambda = 0$. This is the nonpropagating or longitudinal solution. To find the other two, we need to find the eigenvalues of the upper 2×2 block:

$$0 = \det\begin{bmatrix} \kappa - \lambda & i\kappa_g \\ -i\kappa_g & \kappa - \lambda \end{bmatrix}$$

$$= (\kappa - \lambda)^2 - i\kappa_g \cdot (-i\kappa_g)$$

$$= (\kappa - \lambda)^2 - \kappa_g^2.$$

Rearranging,

$$(\kappa - \lambda)^2 = \kappa_g^2$$

$$(\kappa - \lambda) = \pm \kappa_g$$

$$\Rightarrow \lambda = \underline{\underline{\kappa \pm \kappa_g}} \quad \Rightarrow \quad \frac{\omega^2\mu}{k^2} = \kappa \pm \kappa_g$$

The eigenvectors can be found by solving

$$\begin{bmatrix} \kappa - (\kappa \pm \kappa_g) & i\kappa_g \\ -i\kappa_g & \kappa - (\kappa \pm \kappa_g \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = 0$$

Top row: $\rightarrow$ $\{\kappa - (\kappa \pm \kappa_g)\} D_1 + i\kappa_g D_2 = 0$

$$\mp \kappa_g D_1 + i\kappa_g D_2 = 0$$

$$\mp D_1 + i D_2 = 0$$

$$\frac{D_2}{D_1} = \mp i$$

$$\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = D_0 \begin{bmatrix} 1 \\ \mp i \end{bmatrix}$$

In vector form,

$$① \quad k^2 = \omega^2\mu \frac{1}{\kappa + \kappa_g}, \quad \vec{D} = D_0(\hat{x} - i\hat{y})e^{ikz}$$

$$② \quad k^2 = \omega^2\mu \frac{1}{\kappa - \kappa_g}, \quad \vec{D} = D_0(\hat{x} + i\hat{y})e^{ikz}$$

The two characteristic waves are RH and LH circularly polarized plane waves. (which is which?) They travel at different phase velocities.

Faraday rotation:

If a linearly polarized plane wave enters a gyromagnetic medium, we must decompose it as a sum of the two characteristic polarizations in order to analyze what happens to it.

If

$$\vec{D} = \hat{x}\, e^{jkz}$$

at the beginning of the material, then

$$\vec{D} = \left(\frac{\hat{x} - i\hat{y}}{2}\right) e^{ik_I z} + \left(\frac{\hat{x} + i\hat{y}}{2}\right) e^{ik_{II} z}$$

within the material, where

$$k_I = \frac{\omega}{\sqrt{V(K + K_g)}}$$

$$k_{II} = \frac{\omega}{\sqrt{V(K - K_g)}}$$

On exiting the medium, the plane wave has the form

$$\vec{D} = \left[\left(\frac{\hat{x} - i\hat{y}}{2}\right) e^{i\phi_I} + \left(\frac{\hat{x} + i\hat{y}}{2}\right) e^{i\phi_{II}}\right] e^{ikz}$$

where $\phi_I = k_I d$, $\phi_{II} = k_{II} d$. Rearranging,

$$\vec{D} = \frac{1}{2}\left[\hat{x}\left(e^{i\phi_I} + e^{i\phi_{II}}\right) + i\hat{y}\left(e^{i\phi_I} + e^{i\phi_{II}}\right)\right] e^{ikz}$$

The ratio of D_y to D_x is

$$\frac{D_y}{D_x} = \frac{i\left(e^{i\phi_{II}} - e^{i\phi_I}\right)}{\left(e^{i\phi_{II}} + e^{i\phi_I}\right)} = i\,\frac{e^{i(\phi_{II}+\phi_I)/2}\left(e^{i(\phi_{II}-\phi_I)/2} - e^{-i(\phi_{II}-\phi_I)/2}\right)}{e^{i(\phi_{II}+\phi_I)/2}\left(e^{i(\phi_{II}-\phi_I)/2} + e^{-i(\phi_{II}-\phi_I)/2}\right)}$$

$$= i\,\frac{2i\sin\left(\frac{\phi_{II}-\phi_I}{2}\right)}{2\cos\left(\frac{\phi_{II}-\phi_I}{2}\right)}$$

$$= -\tan\left(\frac{\phi_{II}-\phi_I}{2}\right)$$

Since the components are in phase, the wave is linearly polarized, but at a rotated angle $\frac{\phi_{II} - \phi_I}{2}$.

Consider a general linear medium with

$$\bar{D} = \bar{\bar{\epsilon}}\cdot\bar{E} + \bar{\bar{\xi}}\cdot\bar{H}$$
$$\bar{B} = \bar{\bar{\mu}}\cdot\bar{H} + \bar{\bar{\zeta}}\cdot\bar{E}$$

Maxwell's equations become

$$\nabla\times\bar{E} = i\omega\bar{B} = i\omega[\bar{\bar{\mu}}\cdot\bar{H} + \bar{\bar{\zeta}}\cdot\bar{E}] = i\omega\bar{\bar{\mu}}\cdot\bar{H} + i\omega\bar{\bar{\zeta}}\cdot\bar{E}$$
$$\nabla\times\bar{H} = -i\omega\bar{D} = -i\omega[\bar{\bar{\epsilon}}\cdot\bar{E} + \bar{\bar{\xi}}\cdot\bar{H}] = -i\omega\bar{\bar{\epsilon}}\cdot\bar{E} - i\omega\bar{\bar{\xi}}\cdot\bar{H}$$

If we define $\bar{\bar{k}}$ such that

$$\bar{\bar{k}}\cdot\bar{A} = \bar{k}\times\bar{A} = \begin{bmatrix} k_y A_z - k_z A_y \\ k_z A_x - k_x A_z \\ k_x A_y - k_y A_x \end{bmatrix}$$

then

$$\bar{\bar{k}} = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix}$$

Maxwell's equations for a plane wave become

$$i\bar{\bar{k}}\cdot\bar{E} = i\omega\bar{\bar{\mu}}\cdot\bar{H} + i\omega\bar{\bar{\zeta}}\cdot\bar{E}$$
$$i\bar{\bar{k}}\cdot\bar{H} = -i\omega\bar{\bar{\epsilon}}\cdot\bar{E} - i\omega\bar{\bar{\xi}}\cdot\bar{H}$$

We now want to eliminate $\bar{H}$; From Faraday's law,

$$\bar{H} = \frac{1}{i\omega}\bar{\bar{\mu}}^{-1}\cdot[i\bar{\bar{k}}\cdot\bar{E} - i\omega\bar{\bar{\zeta}}\cdot\bar{E}] = \frac{1}{i\omega}\bar{\bar{\mu}}^{-1}\cdot[i\bar{\bar{k}} - i\omega\bar{\bar{\zeta}}]\cdot\bar{E}$$

Substituting into Ampere's law,

$$i\bar{\bar{k}}\cdot\frac{1}{i\omega}\bar{\bar{\mu}}^{-1}[i\bar{\bar{k}} - i\omega\bar{\bar{\zeta}}]\cdot\bar{E} = -i\omega\bar{\bar{\epsilon}}\cdot\bar{E} - i\omega\bar{\bar{\xi}}\cdot\frac{1}{i\omega}\bar{\bar{\mu}}^{-1}\cdot[i\bar{\bar{k}} - i\omega\bar{\bar{\zeta}}]\cdot\bar{E}$$

or

$$\left\{[i\bar{\bar{k}} + i\omega\bar{\bar{\xi}}]\cdot\frac{1}{i\omega}\bar{\bar{\mu}}^{-1}[i\bar{\bar{k}} - i\omega\bar{\bar{\zeta}}] + i\omega\bar{\bar{\epsilon}}\right\}\cdot\bar{E} = 0$$

Rearranging slightly, we have

$$\left\{ \left[\bar{\bar{k}} + \omega\, \bar{\bar{\xi}}\right] \cdot \bar{\bar{\mu}}^{-1} \cdot \left[\bar{\bar{k}} - \omega\, \bar{\bar{\zeta}}\right] + \omega^2 \bar{\bar{\epsilon}} \right\} \cdot \bar{E} = 0 \qquad (*)$$

In order for this equation to have a nontrivial solution, we must have

$$\det \left\{ \left[\bar{\bar{k}} + \omega\, \bar{\bar{\xi}}\right] \cdot \bar{\bar{\mu}}^{-1} \cdot \left[\bar{\bar{k}} - \omega\, \bar{\bar{\zeta}}\right] + \omega^2 \bar{\bar{\epsilon}} \right\} = 0$$

Given the direction of propagation, the only unknown is $k = \| E \|$, so this can be solved for the possible values of k:

$$\det \left\{ \left[\frac{\bar{\bar{k}}}{k} + \frac{\omega}{k} \bar{\bar{\xi}}\right] \cdot \bar{\bar{\mu}}^{-1} \cdot \left[\frac{\bar{\bar{k}}}{k} - \frac{\omega}{k} \bar{\bar{\zeta}}\right] + \frac{\omega^2}{k^2} \bar{\bar{\epsilon}} \right\} = 0$$

= polynomial in k $\Rightarrow$ roots are possible values of k.

The corresponding field directions can be found by substituting these values into $(*)$ and solving for $\bar{E}$.