In a vacuum, $\vec{E}, \vec{D}$ and $\vec{H}, \vec{B}$ are related by proportionality constants ϵ_0 and μ_0. In materials, charged particles respond to applied fields and change the field. We model this effect by changing the relationship between the field quantities.

Dielectric

$$\vec{D} = \epsilon_0 \vec{E} + \underbrace{\vec{P}}_{\text{additional flux due to molecular dipoles}}, \quad \vec{P} = \chi_0 \vec{E}$$

$$= \epsilon \vec{E}, \quad \epsilon \neq \epsilon_0, \quad \epsilon/\epsilon_0 = \epsilon_r \text{ - relative permittivity}$$

Magnetic material

$$\vec{B} = \mu_0 \vec{H} + \mu_0 \underbrace{\vec{M}}_{\text{magnetization}}, \text{ proportional to } \vec{H}.$$

$$= \mu \vec{H}, \quad \mu \neq \mu_0$$

Nonlinear

$$\vec{D} = \epsilon(\vec{E})\, \vec{E}$$

Temporally dispersive

$$\vec{D} = \epsilon(\omega)\, \vec{E}$$

Spatially dispersive

$$\vec{D} = \epsilon(\hat{k})\, \vec{E}$$

Inhomogeneous

$$\vec{D} = \epsilon(\vec{r})\, \vec{E}$$

Anisotropic

$$\vec{D} = \bar{\bar{\epsilon}}\, \vec{E} \rightarrow \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

Anisotropic means 'not isotropic', and isotropic means 'the same in all directions. For an isotropic medium, the relationship between D and E is the same regardless of the direction of $\vec{E}$:

$$\vec{D} = \epsilon \vec{E} \Rightarrow \begin{bmatrix} \epsilon & & \\ & \epsilon & \\ & & \epsilon \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \qquad \text{(Isotropic)}$$

For an anisotropic material, the relationship is direction-dependent:

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \quad \text{or} \quad \vec{D} = \bar{\bar{\epsilon}} \vec{E}$$

In tensor or index notation,

$$D_i = \epsilon_{ij} E_j \equiv \sum_{j=1}^{3} \epsilon_{ij} E_j$$

Of course, there are also magnetically anisotropic materials:

$$\vec{B} = \bar{\bar{\mu}} \vec{H}$$

We also allow for bianisotropic media:

$$\vec{E} = \bar{\bar{\kappa}} \cdot \vec{D} + \bar{\bar{\chi}} \cdot \vec{B} \qquad (\bar{\bar{\kappa}} = \bar{\bar{\epsilon}}^{-1} \text{ if } \bar{\bar{\chi}} = 0)$$

$$\vec{H} = \bar{\bar{\gamma}} \cdot \vec{D} + \bar{\bar{\nu}} \cdot \vec{B}$$

We want to analyze wave propagation in an electrically anisotropic medium:

$$\vec{D} = \bar{\bar{\epsilon}} \vec{E} \qquad \text{(matrix)}$$

$$\vec{B} = \mu \vec{H} \qquad \text{(scalar)}$$

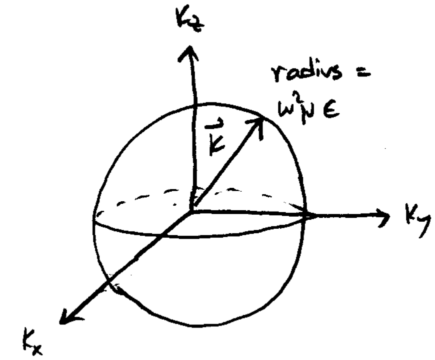
We will use the method of separation of variables, as in the isotropic case. Recall that in the isotropic case, we assume an E field of the form

$$\vec{E}(\vec{r}) = \vec{E}_o e^{i\vec{k}\cdot\vec{r}} = \vec{E}_o e^{ik_x x + ik_y y + ik_z z}$$

where k_x, k_y, and k_z are unknown. The goal of the analysis is to find constraints on $\vec{E}_o$ and $\vec{k}$ such that $\vec{E}(\vec{r})$ is a solution to Maxwell's equations. In the isotropic case, the solution is

$$\left.\begin{aligned} k^2 \equiv k_x^2 + k_y^2 + k_z^2 &= \omega^2 \mu \epsilon \\ \vec{k} \cdot \vec{E}_o &= 0 \end{aligned}\right\} \Rightarrow \text{plane wave solution}$$

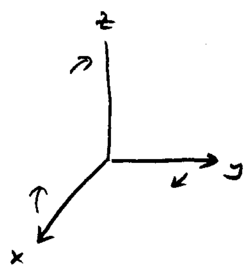
If we define a "k-space" of possible $\vec{k}$ vectors, we find that for a valid plane wave, $\vec{k}$ lies on a sphere of radius $\omega^2\mu\epsilon$:



How does the solution change for an anisotropic medium? We will analyze this, but first let's look at some special cases to get some physical insight.

<u>On-axis propagation</u>

For many matrices, there are coordinate axes in which the matrix is diagonal. Thus, if we rotate the coordinate system while leaving the anisotropic material fixed, we have



$$\bar{\bar{\epsilon}} = \begin{bmatrix} \epsilon_x & 0 & 0 \\ 0 & \epsilon_y & 0 \\ 0 & 0 & \epsilon_z \end{bmatrix}$$

Now, suppose that a plane wave propagates in the z direction, and is polarized in the x direction:

$$\vec{E} = \hat{x}\, e^{ikz}$$

What is k? If we put this field into the constitutive relation,

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_x & & \\ & \epsilon_y & \\ & & \epsilon_z \end{bmatrix} \begin{bmatrix} e^{ikz} \\ 0 \\ 0 \end{bmatrix} \Rightarrow \vec{D} = \hat{x}\, \epsilon_x e^{ikz}$$

If we then apply Faraday's and Ampere's law, we find that

$$k^2 = \omega^2\mu\epsilon_x$$

If the wave had been y-polarized, then we would obtain $k^2 = \omega^2\mu\epsilon_y$.

We find that the phase speed of the wave depends on the polarization.

What if $\vec{E} = (E_{0x}\,\hat{x} + E_{0y}\,\hat{y})\, e^{ikz}$? What is the phase speed of the wave? Physical intuition says that it might be between the two values

$$\omega\sqrt{\mu\epsilon_x} < k < \omega\sqrt{\mu\epsilon_y}$$

(if $\epsilon_x < \epsilon_y$), depending on the angle of $\vec{E}_0$ in the x-y plane. To determine k, we need a more rigorous analysis for this off-axis wave.

<u>Orthogonality relations</u>

In free space, $\vec{E}$ and $\vec{H}$ are perpendicular to $\vec{D}$ and $\vec{B}$, and all the fields are perpendicular to the direction of propagation $\vec{k}$.

Some of these relations hold in an anisotropic medium. If the field spatial dependence is of the form

$$e^{i\vec{k}\cdot\vec{r}}$$

then Maxwell's equations simplify to

$$\vec{k}\times\vec{E} = \omega\vec{B}$$

$$\vec{k}\times\vec{H} = -\omega\vec{D}$$

$$\vec{k}\cdot\vec{B} = 0$$

$$\vec{k}\cdot\vec{D} = 0$$

so that $\vec{k}\perp D$, $\vec{k}\perp B$, and from $\vec{S} = \vec{E}\times\vec{H}^*$, $\vec{S}\perp\vec{E}$ and $\vec{S}\perp\vec{H}$. But as we will see, $\vec{E}$ is not necessarily perpendicular to $\vec{k}$ in an anisotropic medium, and other orthogonality relations valid in free space no longer hold as well.

We will now perform a more rigorous analysis of propagation in electrically anisotropic media:

$$\vec{D} = \bar{\bar{\epsilon}}\vec{E} \quad \text{or} \quad \vec{E} = \bar{\bar{\kappa}}\vec{D}, \quad \bar{\bar{\kappa}} = \bar{\bar{\epsilon}}^{-1}$$

$$\vec{B} = \mu\vec{H}$$

The goal is to take a plane wave of the form

$$\vec{E}(\vec{r}) = \vec{E}_0 e^{i\vec{k}\cdot\vec{r}} \qquad (1)$$

where the direction $\hat{k} = \vec{k}/k$ is given and determine two things:

Goal: find $\begin{cases} \text{① magnitude } k = |\vec{k}| \\ \text{② Possible directions of } \vec{E}_0 \text{ (polarizations)} \end{cases}$

The basic idea is to substitute (1) into Maxwell's equations. We start by taking the curl of Faraday's law:

$$\nabla\times(\nabla\times\vec{E}) = i\omega\mu(\nabla\times\vec{H})$$

Substituting Ampere's law,

$$\nabla\times\nabla\times\vec{E} = i\omega\mu\left(-i\omega\bar{\bar{\epsilon}}\vec{E}\right) = \omega^2\mu\bar{\bar{\epsilon}}\vec{E}$$

We now use the identity $-\nabla\times\nabla\times\vec{E} + \nabla(\nabla\cdot\vec{E}) = \nabla^2\vec{E}$:

$$\nabla^2\vec{E} - \nabla(\nabla\cdot\vec{E}) + \omega^2\mu\bar{\bar{\epsilon}}\vec{E} = 0 \qquad (2)$$

Unlike the free space case, $\nabla\cdot\vec{E} \neq 0$, since $\nabla\cdot\vec{D} = 0$ but $\nabla\cdot(\bar{\bar{\epsilon}}\vec{E}) \neq \bar{\bar{\epsilon}}\nabla\cdot\vec{E}$, so we must retain the divergence term.

By Eq. (1), the del operator becomes

$$\nabla \rightarrow i\vec{k}$$

Using this in (2) gives

$$(i\vec{k})\cdot(i\vec{k})\,\vec{E} - i\vec{k}(i\vec{k}\cdot\vec{E}) + \omega^2\mu\bar{\bar{\epsilon}}\vec{E} = 0$$

The second term can be written in matrix form using

$$\nabla(\nabla\cdot\vec{E}_0 e^{i\vec{k}\cdot\vec{r}}) = i\vec{k}(i\vec{k}\cdot\vec{E})$$

$$= -\vec{k}\,\vec{k}\cdot\vec{E}$$

$$= \underbrace{\begin{bmatrix} k_x^2 & k_xk_y & k_xk_z \\ k_xk_y & k_y^2 & k_yk_z \\ k_xk_z & k_yk_z & k_z^2 \end{bmatrix}}_{\vec{k}\vec{k}^T} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

Using this result, we have

$$-k^2\vec{E} + \vec{k}\vec{k}^T\vec{E} + \omega^2\mu\bar{\bar{\epsilon}}\vec{E} = 0$$

$$(-k^2\bar{\bar{I}} + \vec{k}\vec{k}^T + \omega^2\mu\bar{\bar{\epsilon}})\vec{E} = 0$$

Using $\vec{E} = \bar{\bar{\kappa}}\vec{D}$ and $\nu = 1/\mu$,

$$(-k^2\bar{\bar{I}} + \vec{k}\vec{k}^T + \omega^2\tfrac{1}{\nu}\bar{\bar{\epsilon}})\bar{\bar{\kappa}}\vec{D} = 0$$

$$\left[\nu(-k^2\bar{\bar{I}} + kk^T)\bar{\bar{\kappa}} + \omega^2\right]\vec{D} = 0$$

Using $\vec{k} = k\hat{k}$, we have finally

$$\left[\left(\frac{\omega^2}{k^2}\right)\bar{\bar{I}} - \nu(\bar{\bar{I}} - \hat{k}\hat{k}^T)\bar{\bar{\kappa}}\right]\vec{D} = 0$$

This is an eigenvalue/eigenvector equation!

$$[\lambda\bar{\bar{I}} - \vec{A}]\vec{x} = 0$$

where $\lambda = \omega^2/k^2$, $\vec{A} = \nu(\bar{\bar{I}} - \hat{k}\hat{k}^T)\bar{\bar{\kappa}}$, and $\vec{x} = \vec{D}$. The eigenvalues give us k, and the eigenvectors give us the possible polarizations. We call these eigen<u>waves</u> or <u>characteristic</u> waves.

Free space

For fun, let's look at this in free space:

$$\left[(\omega^2/k^2)\overline{\overline{I}} - \frac{1}{\mu_0\epsilon_0}(\overline{\overline{I}} - \hat{k}\hat{k}) \right] \vec{D} = 0$$

The matrix $\overline{\overline{I}} - \hat{k}\hat{k}$ is a rank-one perturbation of the identity, and its eigenvalues and eigenvectors are easy to find:

$\vec{D} \perp \hat{k}$

$$\frac{1}{\mu_0\epsilon_0}(\overline{\overline{I}} - \hat{k}\hat{k})\vec{D} = \frac{1}{\mu_0\epsilon_0}(\overline{\overline{I}})\vec{D}$$

$$= \frac{1}{\mu_0\epsilon_0}\vec{D}$$

$$\text{Eigenvalue} = \frac{1}{\mu_0\epsilon_0} \Rightarrow \frac{\omega^2}{k^2} = \frac{1}{\mu_0\epsilon_0} \Rightarrow \boxed{k^2 = \omega^2\mu_0\epsilon_0}$$

This is a rank two subspace, or an eigenvalue of multiplicity two.

$\vec{D} \parallel \hat{k}$

$$\frac{1}{\mu_0\epsilon_0}(\overline{\overline{I}} - \hat{k}\hat{k})\vec{D} = \frac{1}{\mu_0\epsilon_0}(\vec{D} - \underbrace{\hat{k}(\vec{D}\cdot\hat{k})}_{\vec{D}})$$

$$= \frac{1}{\mu_0\epsilon_0}(\vec{D} - \vec{D})$$

$$= 0$$

$$\text{Eigenvalue} = 0 \Rightarrow \frac{\omega^2}{k^2} = 0 \Rightarrow \boxed{\omega = 0 \ (DC)}$$

This is a longitudinal wave, which violates Maxwell's equation except in the (trivial) static case.

$\Downarrow$

Plane wave in free space

1. $k^2 = \omega^2\mu_0\epsilon_0$
2. $\vec{D}, \vec{E}$ perpendicular to direction of propagation