Green's Theorem in Electromagnetic Field Theory

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***Abstract*— Green's theorems are commonly viewed as integral identities, but they can also be formulated within a more general operator theoretic framework. The radiation integral for fields in terms of a source and Green's function can be derived in this way. We show that Green's theorem can also be used to obtain conservation of energy, the uniqueness, reciprocity, and extinction theorems, Huygen's principle, and a condition satisfied by fields and sources in a lossless, nonradiating system which parallels the definition of reciprocity. Both three-dimensional and two-dimensional problems are considered.**

I. Introduction

In their usual formulation, Green's theorems are presented as identities in connection with integrals of products. Within the more general setting of functional analysis, Green's theorems can also be viewed as relationships involving the formal adjoint of a partial differential operator [1]. In this paper, we develop operator theoretic formulations of Green's theorems for electromagnetic fields, and use these results to consider some basic principles and theorems of electromagnetic field theory from a new point of view.

The operator Green's theorem has a close relationship with the radiation integral and Huygens' principle, reciprocity, energy conservation, lossless conditions, and uniqueness. Many benefits arise from considering these principles using operator Green's theorems.

The typical application of an operator formulation of Green's theorem in partial differential equation theory is in deriving an integral representation for the solution in terms of given sources or forcing functions. We use this procedure to recover the usual electromagnetic radiation integral, but with the interesting twist that the volume integral term is obtained together with the Huygens principle or surface integral terms in a combined expression. This result sheds light on the subtle connection between Huygens' principle and the extinction theorem.

We also demonstrate that using operator Green's theorems, reaction and energy integrals can be treated on the same footing, leading to parallel expressions for energy conservation and reciprocity relationships. Considering the implications of using either an inner product or a reaction integral in formulating the operator Green's theorem helps to differentiate the often-misunderstood symmetry and self-adjointness properties for Maxwell's equations. As expected, operator symmetry is associated is associated with reciprocity, and self-adjointness is associated with lossless systems, although for a given boundary value problem, the symmetry and self-adjointness properties are only formal unless the fields satisfy certain types of boundary conditions. The uniqueness theorem can also be obtained a way that clarifies the assumption of nonzero loss for uniqueness in the time-harmonic Maxwell boundary value problem and the meaning of uniqueness in the ideal, lossless case.

In this paper, the exterior calculus and differential forms notation are used. These mathematical methods are the subject of many books and papers (*c.f.* [2], [3] and the references therein). We consider time-harmonic fields with $e^{j\omega t}$ time dependence.

II. Green's Theorems in Three Dimensions

In this section, we will first present Green's first and second theorems theorems for three-dimensional fields as identities. We will then develop a new formulation of Green's theorem for electromagnetic fields from an operator theoretic point of view, and apply the result in reconsidering several principles and theorems of field theory.

A. Green's Theorems as Identities

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be one-forms that are continuous together with their first and second derivatives in the volume V and on the boundary S. With Stokes' theorem we obtain

$$\int_V \mathrm{d}(\mathcal{E}_1 \wedge \star\mathrm{d}\,\mathcal{E}_2) = \int_S \mathcal{E}_1 \wedge \star\mathrm{d}\,\mathcal{E}_2\,. \tag{1}$$

where $\star$ is the Hodge star operator and $\wedge$ is the exterior product. Expanding the differential form at the left-hand side into

$$\mathrm{d}\,(\mathcal{E}_1 \wedge \star\mathrm{d}\,\mathcal{E}_2) = \mathrm{d}\,\mathcal{E}_1 \wedge \star\mathrm{d}\,\mathcal{E}_2 - \mathcal{E}_1 \wedge \mathrm{d}\star\mathrm{d}\,\mathcal{E}_2 \tag{2}$$

yields Green's first vector theorem,

$$\int_V (\mathrm{d}\,\mathcal{E}_1 \wedge \star\mathrm{d}\,\mathcal{E}_2 - \mathcal{E}_1 \wedge \mathrm{d}\star\mathrm{d}\,\mathcal{E}_2) = \int_S \mathcal{E}_1 \wedge \star\mathrm{d}\,\mathcal{E}_2 \tag{3}$$

Interchanging $\mathcal{E}_1$ and $\mathcal{E}_2$ yields

$$\int_V (\mathrm{d}\,\mathcal{E}_1 \wedge \star\mathrm{d}\,\mathcal{E}_2 - \mathcal{E}_2 \wedge \mathrm{d}\star\mathrm{d}\,\mathcal{E}_1) = \int_S \mathcal{E}_2 \wedge \star\mathrm{d}\,\mathcal{E}_1 \tag{4}$$

Subtracting (4) from (3) we obtain Green's second vector theorem,

$$\int_V (\mathcal{E}_2 \wedge \mathrm{d}\star\mathrm{d}\,\mathcal{E}_1 - \mathcal{E}_1 \wedge \mathrm{d}\star\mathrm{d}\,\mathcal{E}_2) = \int_S (\mathcal{E}_1 \wedge \star\mathrm{d}\,\mathcal{E}_2 - \mathcal{E}_2 \wedge \star\mathrm{d}\,\mathcal{E}_1) \tag{5}$$

B. Green's Theorem in Operator Theoretic Setting

Basic to the operator viewpoint on Green's theorem is an inner product defined on the space of interest. Accordingly, we first define an inner product on complex-valued 1-forms u and v over a finite region V as

$$\langle u, v\rangle = \int_V u^* \wedge \star v \tag{6}$$

where the superscript * denotes complex conjugation.

The presence of the complex conjugate is important, for on complex-valued fields it renders the integral in (6) a true inner product. We will later see that the complex conjugate can be removed, in which case the integral is no longer an inner product, but becomes the reaction between a field and current. Different forms of the operator theoretic Green's theorem can be developed using the two formulations, the former with application to conservation of energy and uniqueness, and the latter to the radiation integral, Huygens' principle, and reciprocity relationships.

From Maxwell's equations and the constitutive relations, it is easy to show for a homogeneous, isotropic medium that

$$L\mathcal{E} = j\omega\mu\star\mathcal{J} \tag{7}$$

where $\mathcal{E}$ is the electric field intensity 1-form, $\mathcal{J}$ is the electric current density 2-form, and the partial differential operator is

$$L = -\star d\star d + k^2 \tag{8}$$

The constant k is $\omega\sqrt{\mu\epsilon}$. We allow the medium to be possibly (but not necessarily) lossy, in which case k is complex.

An operator Green's theorem is derived from the definition of formal operator adjoint with respect to an inner product over a region of interest:

$$\langle \mathcal{E}_1, L\mathcal{E}_2\rangle - \langle L^a\mathcal{E}_1, \mathcal{E}_2\rangle = \oint_S P(\mathcal{E}_1, \mathcal{E}_2) \tag{9}$$

where S is the boundary of V and the conjunct $P(\mathcal{E}_1, \mathcal{E}_2)$ depends on the operator L. L^a is called the formal adjoint of L. It may not be the actual adjoint of L, because L and L^a may have different domains and ranges and because the boundary integral term on the right may not be zero. Equation (9) defines the formal adjoint, because a different operator in place of the correct formal adjoint L^a would not lead to a pure surface integral term on the right hand side (*i.e.*, a volume integral would remain).

We now derive an expression for $P(\mathcal{E}_1, \mathcal{E}_2)$ from the definition of the operator L. We make use of the identity

$$\star\alpha \wedge \beta = \alpha \wedge \star\beta \tag{10}$$

for 1-forms α and β. We will also use a multidimensional integration by parts theorem, which is

$$\int_V d\alpha \wedge \beta = \int_V \alpha \wedge d\beta + \oint_S \alpha \wedge \beta \tag{11}$$

where α and β are again 1-forms. This result can be derived from the product rule for the exterior derivative and Stokes' theorem.

We now wish to show that the formal adjoint of L is $L^a = -\star d\star d + {k^*}^2$, by substituting L and L^a into the left hand side of Eq. (9) and showing by computation that a surface integral of the form of the right hand side results. Making the substitutions, we obtain

$$\oint_S P = \int_V \left[\mathcal{E}_1^* \wedge (-\star d\star d + k^2)\mathcal{E}_2 - (-\star d\star d + k^2)\mathcal{E}_1^* \wedge \mathcal{E}^2\right] \tag{12}$$

By the identity (10), the terms containing k^2 are equal and opposite in sign. We integrate the remaining two terms by parts, to obtain

$$\oint_S P = \int_V \left[\star d\mathcal{E}_1^* \wedge d\mathcal{E}_2 - d\mathcal{E}_1^* \wedge \star d\mathcal{E}_2\right] + \oint_S \left[\star d\mathcal{E}_1^* \wedge \mathcal{E}_2 + \mathcal{E}_1^* \wedge \star d\mathcal{E}_2\right] \tag{13}$$

The integrand of the volume term vanishes, also by (10). This leads to the desired operator Green's theorem,

$$\langle \mathcal{E}_1, L\mathcal{E}_2\rangle - \langle L^a\mathcal{E}_1, \mathcal{E}_2\rangle = \oint_S \left[\star d\mathcal{E}_1^* \wedge \mathcal{E}_2 + \mathcal{E}_1^* \wedge \star d\mathcal{E}_2\right] \tag{14}$$

which is essentially Green's second vector theorem (5). Using Faraday's Law, and assuming that there are no sources on S, the boundary integrand can also be written as

$$P = j\omega\mu \left[\mathcal{H}_1^* \wedge \mathcal{E}_2 - \mathcal{E}_1^* \wedge \mathcal{H}_2\right] \tag{15}$$

C. Conservation of Energy

We now show that a conservation of energy relationship can be obtained from the operator Green's theorem in Eq. (14). If we consider a lossless medium, for which k is real, then the operator L is self-adjoint. In this case, Eq. (14) becomes

$$\int_V \mathcal{E}_1^* \wedge \mathcal{J}_2 + \mathcal{J}_1^* \wedge \mathcal{E}_2 = \oint_S \mathcal{H}_1^* \wedge \mathcal{E}_2 - \mathcal{E}_1^* \wedge \mathcal{H}_2 \tag{16}$$

If we set $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$ and $\mathcal{J}_1 = \mathcal{J}_2 = \mathcal{J}$, then we obtain

$$\mathrm{Re}\left\{\int_V \mathcal{E} \wedge \mathcal{J}^*\right\} = -\mathrm{Re}\left\{\oint_S \mathcal{E} \wedge \mathcal{H}^*\right\} \tag{17}$$

This is a statement of the conservation of energy. On the left-hand side is the real power supplied by the source to the field, and on the right is the power flowing out of the volume V across its boundary S. This is the real part of Poynting's theorem.

For a lossless system with an impenetrable boundary condition, Green's theorem also can be used to derive an interesting expression that parallels the definition of reciprocity. If a sheet of perfect electric conductor (PEC) is placed along the surface S, or any other type of boundary condition that allows no power to flow out of the volume V, then the right-hand side of (14) vanishes, and we have

$$\int_V \mathcal{E}_1^* \wedge \mathcal{J}_2 + \int_V \mathcal{J}_1^* \wedge \mathcal{E}_2 = 0 \tag{18}$$

This expression can be taken as a definition of a lossless, nonradiating system, in the same way that (24) is the definition of a reciprocal system. Physically, this result means that a system loses no energy if the complex power supplied by source 1 to field 2 is equal to the complex conjugate of the power supplied by source 2 to field 1. As with the definition of reciprocity, $\mathcal{E}_1$ is excited by $\mathcal{J}_1$, and $\mathcal{E}_2$ is excited by $\mathcal{J}_2$, so that the two integrals in (18) do not actually represent the physical radiation of energy, but must be considered to be measurements by nonperturbing test sources.

D. Uniqueness Theorem

We can derive the uniqueness theorem for time-harmonic electromagnetic fields from Green's theorem. Suppose that the tangential components of a solution pair $\mathcal{E}$, $\mathcal{H}$ to Maxwell's equations for a given source are specified on S. Suppose further that there are two solutions to Maxwell's equations in V, $\mathcal{E}_1 = \mathcal{E}$ and $\mathcal{E}_2 = \mathcal{E} + \delta\mathcal{E}$. By the first assumption, $\delta\mathcal{E} = 0$ on S. From Green's theorem, we have that

$$2\mathrm{Re}\left\{\int_V \mathcal{E} \wedge \mathcal{J}^*\right\} + \int_V \mathcal{J}^* \wedge \delta\mathcal{E} = -2\mathrm{Re}\left\{\oint_S \mathcal{E} \wedge \mathcal{H}^*\right\} \tag{19}$$

But because of (17), the first term on the left is equal to the right hand side, so we must necessarily have that

$$\int_V \mathcal{J}^* \wedge \delta\mathcal{E} = 0 \tag{20}$$

This result provides more insight than the usual proof of the uniqueness theorem. Because both solutions satisfy Maxwell's equations, the difference field $\delta\mathcal{E}$ must be a homogeneous solution, such that $L\delta\mathcal{E} = 0$. Equation (20) requires that the source J supply no energy to the homogeneous solution $\delta\mathcal{E}$. Physically, this represents a field solution that can exist without a driving source, or in other words, a nonradiating resonant mode with infinite quality factor. If the source were to supply energy to such a mode, its amplitude would grow without bound, and the steady state assumption we have made in modeling the fields as time-harmonic would be violated.

Mathematically, the time-harmonic Maxwell boundary value problem for a lossless system actually does not have a unique solution, because there may exist undamped transient solutions. The usual way to deal with this difficulty in proving the uniqueness theorem is to postulate a very small loss that eliminates any nonradiating, resonant modes of the system.

E. Lorentz Reciprocity Theorem

We can derive a slightly different Green's theorem using reaction instead of the inner product (6). We begin with

$$\int_V \mathcal{E}_1 \wedge \star L\mathcal{E}_2 - \int_V L^T\mathcal{E}_1 \wedge \star\mathcal{E}_2 = \oint_S Q(\mathcal{E}_1, \mathcal{E}_2) \tag{21}$$

where the operator L^T is the formal transpose of L. For an isotropic medium, we have that $L = L^T$. For complex phasor fields, the volume integrals are not inner products, because of the absence of a complex conjugate on one of the factors of the integrands. Because $\star L\mathcal{E}$ is a current density, these integrals represent reactions.

Proceeding as in the derivation of (15), we obtain

$$\int_V \mathcal{E}_1 \wedge \star L\mathcal{E}_2 - \int_V L\mathcal{E}_1 \wedge \star\mathcal{E}_2 = -j\omega\mu \oint_S \mathcal{H}_1 \wedge \mathcal{E}_2 + \mathcal{E}_1 \wedge \mathcal{H}_2 \tag{22}$$

By making use of $L\mathcal{E} = j\omega\mu\star J$, the volume integral terms reduce to the standard expression for the reaction between fields and sources:

$$\int_V \mathcal{E}_1 \wedge \mathcal{J}_2 - \int_V \mathcal{J}_1 \wedge \mathcal{E}_2 = -\oint_S \mathcal{H}_1 \wedge \mathcal{E}_2 + \mathcal{E}_1 \wedge \mathcal{H}_2 \tag{23}$$

This is the Lorentz reciprocity theorem.

By definition, a medium is said to be reciprocal if the left-hand side of (23) vanishes for all pairs of sources $\mathcal{J}_1$ and $\mathcal{J}_2$, so that

$$\int_V \mathcal{E}_1 \wedge \mathcal{J}_2 - \int_V \mathcal{J}_1 \wedge \mathcal{E}_2 = 0 \tag{24}$$

We will here reproduce a proof that free space is reciprocal. If S is a sphere with large radius and is in the far field of all sources, then the fields satisfy along S the radiation boundary condition

$$\begin{aligned} \mathcal{E} &= \mathcal{E}_0(\theta,\phi)\frac{e^{-jkr}}{4}, \quad dr \lrcorner \mathcal{E}_0 = 0 && (25)\\ \mathcal{H} &= \frac{1}{\eta}\star(dr \wedge \mathcal{E}) && (26) \end{aligned}$$

The boundary integral in (23) becomes

$$-\frac{e^{-jkr}}{\eta r}\oint_S \mathcal{E}_{02} \wedge \star(dr \wedge \mathcal{E}_{01}) - \mathcal{E}_{01} \wedge \star(dr \wedge \mathcal{E}_{02}) \tag{27}$$

Using the identity $\alpha \lrcorner \beta = \star(\star\beta \wedge \alpha)$, this can be expressed as

$$\frac{e^{-jkr}}{\eta r}\oint_S \star\left[\mathcal{E}_{02}\lrcorner(dr \wedge \mathcal{E}_{01}) - \mathcal{E}_{01}\lrcorner(dr \wedge \mathcal{E}_{02})\right] \tag{28}$$

Expanding the interior product across the exterior product using $\alpha\lrcorner(\beta \wedge \gamma) = (\alpha\lrcorner\beta) \wedge \gamma + (-1)^{\deg\beta}\beta \wedge (\alpha\lrcorner\gamma)$ leads to

$$\begin{aligned} \frac{e^{-jkr}}{\eta r}\oint_S \star\big[&(\mathcal{E}_{02}\lrcorner dr) \wedge \mathcal{E}_{01} - dr \wedge (\mathcal{E}_{02}\lrcorner\mathcal{E}_{01}) \\ &-(\mathcal{E}_{01}\lrcorner dr) \wedge \mathcal{E}_{02} + dr \wedge (\mathcal{E}_{01}\lrcorner\mathcal{E}_{02})\big] \quad (29) \end{aligned}$$

The first and third terms of the integrand vanish because of the orthogonality condition in (25). The second and fourth terms are equal and opposite in sign. As a result, the integrand is identically zero on S, and we have that free space is reciprocal according to (24). This proof can easily be generalized to the case of an anisotropic medium, for which reciprocity holds as long as $\star_h = \tilde{\star}_h$ and $\star_e = \tilde{\star}_e$ in the notation of [4].

F. Green's Forms and the Radiation Integral

We finally turn to the most common application of a Green's theorem in the theory of boundary value problems. This is the derivation of an integral solution or radiation integral for a boundary value problem. The transposed double $1 \otimes 1$ tensor Green's function satisfies [5]

$$L^T \mathcal{G}^T(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')\mathcal{I} \tag{30}$$

where $\mathcal{I}$ is the unit double $1 \otimes 1$ form $dx\,dx' + dy\,dy' + dz\,dz'$. The derivative operator acts on the unprimed coordinate $\mathbf{r}$. We define here the formal transpose of the Green's function, rather than the Green's function satisfying $L\mathcal{G}(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')\mathcal{I}$, because for a general medium the radiation integral is given in terms of the formal transpose $\mathcal{G}^T$ rather than $\mathcal{G}$ [4]. For an isotropic medium, $\mathcal{G}^T = \mathcal{G}$, so that there is no distinction between the Green form and its formal transpose. In this paper, we assume that the medium is isotropic.

By making the substitutions $\mathcal{E}_1(\mathbf{r}) = \mathcal{G}(\mathbf{r}, \mathbf{r}')$ and $E_2 = E$ in (22), we obtain the radiation integral

$$\begin{aligned}\mathcal{E}(\mathbf{r}') &= -j\omega\mu \int_V \mathcal{G}(\mathbf{r}, \mathbf{r}') \wedge \mathcal{J}(\mathbf{r}) \\ &\quad + \oint_S \star d\mathcal{G}(\mathbf{r}, \mathbf{r}') \wedge \mathcal{E}(\mathbf{r}) - j\omega\mu\mathcal{G}(\mathbf{r}, \mathbf{r}') \wedge \mathcal{H}(\mathbf{r}) \end{aligned} \tag{31}$$

where the integrations are over the $\mathbf{r}$ coordinate. Using either shift invariance $\mathcal{G}(\mathbf{r}', \mathbf{r}) = \mathcal{G}(|\mathbf{r} - \mathbf{r}'|)$ or more generally the symmetry properties of $\mathcal{G}(\mathbf{r}', \mathbf{r})$, the coordinate dependence of $\mathcal{G}(\mathbf{r}', \mathbf{r})$ can be interchanged and placed in the usual convention for the radiation integral.

This expression is of note because the volume integral term appears together with a surface integration, which is the mathematical expression of Huygens' principle. It is interesting to consider the consequences of this rigorous combination of the two terms. If the fields $\mathcal{E}$ and $\mathcal{H}$ on the boundary S are produced by impressed sources inside the region of interest V, then the field solution $\mathcal{E}$ is given entirely by the volume integral in (31). As a consequence, the surface integral must vanish. This is the extinction theorem. Alternately, if the fields are produced by sources entirely outside of V, then the volume term vanishes, because its integrand is zero over the domain of integration. In this case, the surface integral term becomes nonzero. This is Huygen's principle.

The surface integral in (31) has properties that are both interesting and subtle, but which can be better understood when viewed in conjunction with the volume term. The subtlety arises in that the integrands of the surface integral term in the two cases mentioned above can be made to be almost identical, yet in one case the integral vanishes, and in the other it is nonzero. To consider a concrete example, if V is a spherical ball containing a point source, then the surface term vanishes. If V is the exterior of the ball, then the volume term is zero and the surface term gives the fields radiated by the point source outside the ball. In the two cases, the integrands are nearly identical, yet in one case the result of the integral vanishes, and the other case it is nonzero. The only differences between the two surface integrations are that one case, S is the boundary of a sphere, and in the other S is the boundary of all space except the sphere, so that the orientations for the surface normal direction for S are different, and the field observation points are in different regions.

III. Green's Theorems in Two Dimensions

We now turn to the case of two-dimensional problems, which arise if the properties of transverse modes in electromagnetic structures exhibiting cylindrical symmetry are investigated. For the analysis of two-dimmensional structures the scalar Green's theorems in two dimensions are useful. As before, we first formulate the Green's theorems as usually constituted, and then turn to the operator theoretic setting.

A. Green's Theorems as Identities - 2D

There are two approaches to formulating the 2D Green's theorems. One is to use exterior calculus for a 2D space. This has the advantage that the algebra of differential forms is almost completely unchanged in a formal sense, so that nearly all results appear exactly as they do in a 3D space, but with different degrees for the differential forms and, consequently, different actions of the operators on the differential forms. There is, however, a serious disadvantage to the use of the 2D exterior calculus in this way. The 2D calculus can lead to confusion if used side by side in the same document with the 3D formulation, without great care to point out which expressions are to be interpreted as 2D and which as 3D (the major problem being the Hodge star operator). For this reason, we formulate the 2D version of the Green's theorem here using the 3D exterior calculus.

Consider a structure with general cylindric symmetry. In a Cartesian coordinate system x, y, z let the z-axis be the axis of the cylindric structure and let y and z be the transverse coordinates. The cylindric structure under investigation exhibits translational invariance in the z direction. We introduce the transverse exterior derivative

$$\mathrm{d}_t\mathcal{U} = \mathrm{d}x\frac{\partial\mathcal{U}}{\partial x} + \mathrm{d}y\frac{\partial\mathcal{U}}{\partial y} \tag{32}$$

For a two-form $\mathcal{U}$ exhibiting only transverse components,

$$\mathcal{U}(x, y) = U_x(x, y)\mathrm{d}y \wedge \mathrm{d}z + U_y(x, y)\mathrm{d}z \wedge \mathrm{d}x \tag{33}$$

the Stokes' theorem becomes

$$\oint_S \mathcal{U} = \int_V \mathrm{d}_t\mathcal{U} \tag{34}$$

In component notation this is

$$\oint_S (U_x(x, y)\mathrm{d}y - U_y(x, y)\mathrm{d}x) \wedge \mathrm{d}z \tag{35}$$

$$= \int_V \left(\frac{\partial U_x(x, y)}{\partial x} + \frac{\partial U_x(x, y)}{\partial y}\right) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \tag{36}$$

Since the integrands in both sides of the equation do not depend on z we can omit the integration over z, so that

$$\oint_C (U_x(x, y)\mathrm{d}y - U_y(x, y)\mathrm{d}x) \tag{37}$$

$$= \int_A \left(\frac{\partial U_x(x, y)}{\partial x} + \frac{\partial U_x(x, y)}{\partial y}\right) \mathrm{d}x \wedge \mathrm{d}y \tag{38}$$

On the right-hand side of this equation the integration is performed over a cross-sectional area A of the cylindrical structure and the integral on the left-hand side of the equation is performed over the boundary curve C of the area A. We can write this two-dimensional Stokes' theorem as

$$\oint_C \mathcal{U} \lrcorner \mathrm{d}z = \int_A (\mathrm{d}_t \mathcal{U}) \lrcorner \mathrm{d}z \tag{39}$$

Let $\phi(x, y)$ and $\psi(x, y)$ be two-dimensional scalar functions continuous together with their first and second derivatives in the area A and on the boundary C. With the two-dimensional Stokes' theorem (39) we obtain

$$\int_A \mathrm{d}_t(\star(\psi \mathrm{d}_t\phi)) \lrcorner \mathrm{d}z = \oint_C (\star(\psi \mathrm{d}_t\phi)) \lrcorner \mathrm{d}z \tag{40}$$

Allowing the scalar ψ to pass through the Hodge star operator by linearity in the integrand on the left-hand side allows the product rule for the exterior derivative to be used, to obtain

$$\mathrm{d}_t(\psi \wedge \star \mathrm{d}_t\phi) = (\mathrm{d}_t\psi) \wedge \star \mathrm{d}_t\phi + \psi \wedge \mathrm{d}_t \star \mathrm{d}_t\phi \tag{41}$$

Inserting this result into (40) leads to the two-dimensional form of Green's first scalar theorem,

$$\int_A (\mathrm{d}_t\psi \wedge \star \mathrm{d}_t\phi) \lrcorner \mathrm{d}z + \int_A (\psi\, \mathrm{d}_t \star \mathrm{d}_t\phi) \lrcorner \mathrm{d}z = \int_C (\star(\psi \mathrm{d}\phi)) \lrcorner \mathrm{d}z \tag{42}$$

Let us choose a coordinate system u, n, z such that u and n are the transverse coordinates and z is the longitudinal coordinate. furthermore u is tangential and n is normal to the boundary curve C. This yields

$$(\star(\phi \mathrm{d}_t\psi)) \lrcorner \mathrm{d}z = \psi \frac{\partial \phi}{\partial u} \frac{g_n}{g_u} \mathrm{d}n - \psi \frac{\partial \phi}{\partial n} \frac{g_u}{g_n} \mathrm{d}u \tag{43}$$

Inserting this into (42) and using

$$\mathrm{d}_t \star \mathrm{d}_t\phi = \star \Delta_t \phi = \Delta\phi\, g_u g_n \mathrm{d}u \wedge \mathrm{d}n \wedge \mathrm{d}z \tag{44}$$

where g_u and g_n are metrical coefficients associated with the u, n coordinates, yields an alternative two-dimensional form of Green's first scalar theorem,

$$\int_A \left(\frac{\partial \phi}{\partial u} \frac{\partial \psi}{\partial u} \frac{g_n}{g_u} + \frac{\partial \phi}{\partial n} \frac{\partial \psi}{\partial n} \frac{g_u}{g_n} \right) \mathrm{d}u \wedge \mathrm{d}n + \int_A \psi\, \Delta\phi\, g_u g_n \mathrm{d}u \wedge \mathrm{d}n = - \oint_C \psi \frac{\partial \phi}{\partial n} \frac{g_u}{g_n} \mathrm{d}u \tag{45}$$

Interchanging ϕ and ψ in (42) and forming the difference between both equations, considering that $\mathrm{d}\psi \wedge \star \mathrm{d}\phi = \mathrm{d}\phi \wedge \star \mathrm{d}\psi$, yields the two-dimensional form of Green's second scalar theorem

$$\int_A (\psi\, \mathrm{d}_t \star \mathrm{d}_t\phi - \phi\, \mathrm{d}_t \star \mathrm{d}_t\psi) \lrcorner \mathrm{d}z = \oint_C [\star(\psi \mathrm{d}_t\phi) - \star(\phi \mathrm{d}_t\psi)] \lrcorner \mathrm{d}z \tag{46}$$

Inserting (43) yields an alternative two-dimensional form of Green's second scalar theorem,

$$\int_A (\psi\, \mathrm{d}_t \star \mathrm{d}_t\phi - \phi\, \mathrm{d}_t \star \mathrm{d}_t\psi) \lrcorner \mathrm{d}z = \oint_C \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) \frac{g_u}{g_n} \mathrm{d}u \tag{47}$$

With (44) we obtain from this

$$\int_A (\psi\, \Delta_t\phi - \phi\, \Delta_t\psi) g_u g_n \mathrm{d}u \wedge \mathrm{d}n = \oint_C \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) \frac{g_u}{g_n} \mathrm{d}u \tag{48}$$

The alternate forms of the theorems are useful in that they show explicitly the dependence of the boundary integration on both the scalars and their normal derivatives.

B. Green's Theorem in Operator Theoretic Setting - 2D

We now turn finally to the development of Green's theorem as a statement of a formal operator adjoint condition. In this case, the appropriate inner product is

$$\langle \psi, \phi \rangle = \int_A (\psi^* \star \phi) \lrcorner dz \tag{49}$$

The scalar fields are assumed to satisfy a Helmholtz equation of the form

$$M\phi = (\Delta_t + k^2)\phi = f \tag{50}$$

where $\Delta_t = \star \mathrm{d}_t \star \mathrm{d}_t$ on 0-forms, so that the scalar field could represent, for example, the z component of the electric or magnetic fields.

The formal operator adjoint L^a satisfies

$$\langle \psi, M\phi \rangle - \langle M^a\psi, \phi \rangle = \oint_C R(\psi, \phi) \tag{51}$$

which has the same form as (9) for the 3D case. The formal adjoint M^a with respect to the inner product (49) is simply the complex conjugate of M. Substituting expressions for M and M^a into Eq. (51) and cancelling the terms containing k^2 yields

$$\langle \psi, M\phi \rangle - \langle M^a\psi, \phi \rangle = \int_A \left(\psi^* \mathrm{d}_t \star \mathrm{d}_t \phi - \phi \mathrm{d}_t \star \mathrm{d}_t \psi^* \right) \lrcorner dz \tag{52}$$

Using Eq. (46) leads to the desired form of Green's theorem,

$$\langle \psi, M\phi \rangle - \langle M^a\psi, \phi \rangle = \oint_C [\star(\psi^* \mathrm{d}_t\phi) - \star(\phi \mathrm{d}_t\psi^*)] \lrcorner \mathrm{d}z \tag{53}$$

This expression can be applied in the same way for 2D problems as the corresponding result (14) for 3D fields.

C. Radiation Integral - 2D

For completeness, we develop the integral solution for a two-dimensional boundary value problem consisting of the partial differential equation (50) and a prescribed value for the scalar field on the boundary curve C. The Green's function for this boundary value problem satisfies the definition

$$Mg(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') \tag{54}$$

on the region A as well as the given boundary condition on C. We now make the substitution $\psi(\mathbf{r}) = g^a(\mathbf{r}, \mathbf{r}')$ in Eq. (53), where g^a is defined analogously to (54) but with the adjoint operator $M^a = M^*$, so that $g^a = g^*$. This procedure leads to

$$\phi(\mathbf{r}) = \int_A^{\prime} [g(\mathbf{r}, \mathbf{r}') \star f(\mathbf{r}')] \lrcorner dz - \oint_C^{\prime} [\star(g(\mathbf{r}, \mathbf{r}') \mathrm{d}_t' \phi(\mathbf{r}') - \star(\phi(\mathbf{r}') \mathrm{d}_t' g(\mathbf{r}, \mathbf{r}'))] \lrcorner \mathrm{d}z \tag{55}$$

after interchanging $\mathbf{r}$ and $\mathbf{r}'$. Again, the dependence on the normal derivatives of ϕ and g can be made explicit using Eq. (48).

IV. Conclusion

We have considered Green's theorem from two points of view: as integral identities for derivatives of fields, and as statements of formal operator transpose and adjoint relationships. It is hoped that this treatment will help to emphasize the importance of Green's theorems in electromagnetic field theory, and shed some interesting new light on the radiation integral, Huygens principle, reciprocity, energy conservation, uniqueness, and other principles of electromagnetic field theory.

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